A quasiseparable approach to five-diagonal CMV and companion matrices

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Part 1. Twist transformation. Recurrence relations for polynomials associated to certain five-diagonal matrices

Joint work with **Tom Bella** and **Vadim Olshevsky**.

Unitary Hessenberg and CMV matrices

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Both are related to the same system of polynomials orthogonal on the unit circle!

Companion and Fiedler matrix

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -a_1 & -a_2 & 1 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 0 & -a_5 & 0 & -a_6 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$p(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

Common properties of CMV and Fiedler matrices

➠ five-diagonal matrices having a 'staircase' structure:

$$
\begin{bmatrix}
\star & \star & 0 & & & & & \\
\star & \star & \star & \star & & & & \\
\star & \star & \star & \star & 0 & & & \\
& 0 & \star & \star & \star & \star & & \\
& & \star & \star & \star & \star & 0 & \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix},
\quad
\begin{bmatrix}
\star & \star & \star & & & & & \\
\star & \star & \star & 0 & & & & \\
0 & \star & \star & \star & \star & & & \\
& \star & \star & \star & \star & 0 & & \\
& & 0 & \star & \star & \star & \star & \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

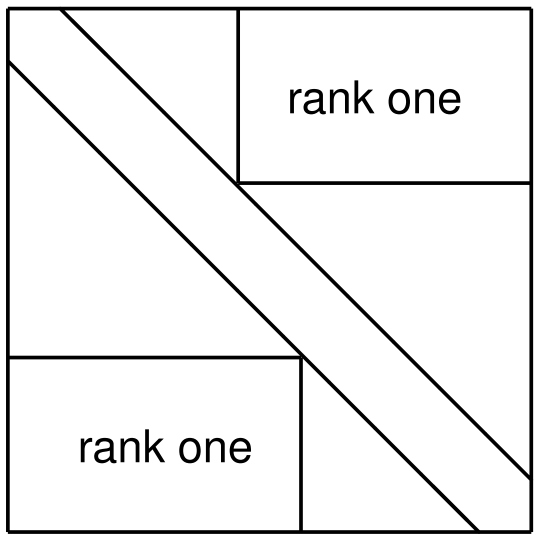
➠ connection to a certain Hessenberg matrix

➠ **quasiseparability**

(1,1)-quasiseparable matrices

Definition. [Rank definition of $(1,1)$–qs matrices]
A matrix A is called $(1,1)$–*qs* (i.e., *Order-One-Quasiseparable*) if

$$\max_{1\leqslant i\leqslant n-1} \operatorname{rank} A(1:i, i+1:n) = \max_{1\leqslant i\leqslant n-1} \operatorname{rank} A(i+1:n, 1:i) = 1.$$



CMV and Fiedler matrices are $(1,1)$–qs

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$F = \begin{bmatrix} -a_1 & -a_2 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & -a_5 & 0 & -a_6 & 1 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

CMV and Fiedler matrices are $(1,1)$–qs

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots \; \ddots \end{bmatrix}$$

$$F = \begin{bmatrix} -a_1 & -a_2 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & -a_5 & 0 & -a_6 & 1 \\ & & & \ddots & \ddots & \ddots & \ddots \; \ddots \end{bmatrix}$$

CMV and Fiedler matrices are $(1,1)$–qs

$$K = \left[\begin{array}{cc|ccccc} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \hline \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots \quad \ddots \end{array}\right]$$

$$F = \left[\begin{array}{cc|ccccc} -a_1 & -a_2 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & \\ \hline 0 & -a_3 & 0 & -a_4 & 1 & & \\ & 1 & 0 & 0 & 0 & 0 & \\ & & 0 & -a_5 & 0 & -a_6 & 1 \\ & & & \ddots & \ddots & \ddots & \ddots \quad \ddots \end{array}\right]$$

CMV and Fiedler matrices are $(1,1)$–qs

$$K = \left[\begin{array}{ccc|ccccc} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\ \hline & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right]$$

$$F = \left[\begin{array}{ccc|ccccc} -a_1 & -a_2 & 1 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & & \\ \hline & 1 & 0 & 0 & 0 & 0 & & \\ & & 0 & -a_5 & 0 & -a_6 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right]$$

CMV and Fiedler matrices are $(1,1)$–qs

$$K = \left[\begin{array}{ccc|ccccc}
-\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\
-\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\
\mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\
\hline
 & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\
 & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right]$$

$$F = \left[\begin{array}{ccc|ccccc}
-a_1 & -a_2 & 1 & & & & & \\
1 & 0 & 0 & 0 & & & & \\
0 & -a_3 & 0 & -a_4 & 1 & & & \\
\hline
 & 1 & 0 & 0 & 0 & 0 & & \\
 & & 0 & -a_5 & 0 & -a_6 & 1 & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{array}\right]$$

Unitary Hessenberg and companion matrices are also $(1,1)$–qs!

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Unitary Hessenberg and companion matrices are also $(1,1)$–qs!

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Unitary Hessenberg and companion matrices are also $(1,1)$–qs!

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Unitary Hessenberg and companion matrices are also $(1,1)$–qs!

$$M = \left[\begin{array}{ccc|cc} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ \hline 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{array}\right]$$

$$C = \left[\begin{array}{ccc|cc} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

Unitary Hessenberg and companion matrices are also $(1,1)$–qs!

$$M = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & -\rho_0^*\mu_1\mu_2\mu_3\rho_4 & -\rho_0^*\mu_1\mu_2\mu_3\mu_4\rho_5 \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & -\rho_1^*\mu_2\mu_3\rho_4 & -\rho_1^*\mu_2\mu_3\mu_4\rho_5 \\ 0 & \mu_2 & -\rho_2^*\rho_3 & -\rho_2^*\mu_3\rho_4 & -\rho_2^*\mu_3\mu_4\rho_5 \\ 0 & 0 & \mu_3 & -\rho_3^*\rho_4 & -\rho_3^*\mu_4\rho_5 \\ 0 & 0 & 0 & \mu_4 & -\rho_4^*\rho_5 \end{bmatrix}$$

$$C = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

A generator representation for $(1,1)$–qs matrices

A matrix is $(1,1)$–qs if and only if it has the following representation

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

where $d_k, q_k, a_k, p_k, g_k, b_k, h_k$ are scalars.

Generators of Unitary Hessenberg and CMV matrices

Table 1: Generators of unitary Hessenberg matrix

d_k	a_k	b_k	q_k	g_k	p_k	h_k
$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k

Table 2: Generators of CMV matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
odd	$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k
even	$-\rho_{k-1}^*\rho_k$	μ_k	0	$-\rho_{k-1}^*\mu_k$	μ_k	ρ_k	1

Generators are interchanged for even indices!

Generators of companion and Fidler matrices

Table 3: Generators of companion matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k = 1$	$-a_1$	$-$	$-$	1	1	$-$	$-$
$k \neq 1$	0	0	1	1	0	1	$-a_k$

Table 4: Generators of Fiedler matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k = 1$	$-a_1$	$-$	$-$	1	1	$-$	$-$
$k > 1$ odd	0	1	0	0	1	$-a_k$	1
even	0	0	1	1	0	1	$-a_k$

Generators are interchanged for odd indices greater than one!

Recurrence relations for polynomials related to $(1,1)$-qs matrices

Theorem. (due to Eidelman, Gohberg and Olshevsky)
Let $\{r_k(x)\}_{k=0}^{n}$ be a system of characteristic polynomials of leading submatrices of an $(1,1)$-qs matrix A. Then they satisfy two-term recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} a_k b_k x - c_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix},$$

where $c_k = d_k a_k b_k - q_k p_k b_k - g_k h_k a_k$.

Corollary 1. The following operation on generators preserve characteristic polynomials of leading submatrices:

TWIST TRANSFORMATION

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

Corollary 2. For every $n \times n$ $(1,1)$–qs matrix there are 2^n (probably not distinct) matrices with the same system of characteristic polynomials.

Hessenberg Order-One quasiseparable matrices

Since both CMV and Fidler matrices were obtained from strongly Hessenberg quasiseparable matrices we define general $(H,1)$–qs matrices

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ 0 & q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ 0 & 0 & q_3 & d_4 & g_4h_5 \\ 0 & 0 & 0 & q_4 & d_5 \end{bmatrix}$$

Remark. Comparing definitions of $(1,1)$–qs and $(H,1)$–qs matrices one can easily see that an $(1,1)$–qs matrix is $(H,1)$–qs if and only if it has a choice of generators such that $a_k = 0$, $p_k = 1$, $q_k \neq 0$.

Twisted $(H, 1)$–qs matrices and their patterns

Definition. [Twisted $(H, 1)$-qs matrices]
An $(1, 1)$-qs matrix A is called **twisted** $(H, 1)$-qs if it can be obtained from an $(H, 1)$-qs matrix via twist transformations:

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

Definition. [Pattern of twisted $(H, 1)$-qs matrices]
For an arbitrary twisted $(H, 1)$-qs matrix A we will say that a sequence of binary digits $(i_1, i_2, \ldots, i_n)$ is the **pattern** of A if A can be transformed to some $(H, 1)$-qs matrix H applying the twist transformations for all $k = i_j$ such that $i_j = 1$. Under these conditions we will also say that $A = H(i_1, i_2, \ldots, i_n)$.

Examples of twisted $(H,1)$–qs matrices

$H(0,0,0,0)$

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 \\ q_1 & d_2 & g_2h_3 & g_2b_3h_4 \\ 0 & q_2 & d_3 & g_3h_4 \\ 0 & 0 & q_3 & d_4 \end{bmatrix}$$

$H(0,1,0,0)$

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 \\ \boxed{h_2}\,q_1 & d_2 & \boxed{q_2}\,h_3 & \boxed{q_2}\,b_3h_4 \\ \boxed{b_2}\,q_1 & \boxed{g_2} & d_3 & g_3h_4 \\ 0 & 0 & q_3 & d_4 \end{bmatrix}$$

$H(1,1,1,1)$

$$\begin{bmatrix} d_1 & q_1 & 0 & 0 \\ g_1h_2 & d_2 & q_2 & 0 \\ g_1b_2h_3 & g_2h_3 & d_3 & q_3 \\ g_1b_2b_3h_4 & g_2b_3h_4 & g_3h_4 & d_4 \end{bmatrix}$$

$H(0,1,0,1)$

$$\begin{bmatrix} d_1 & g_1 & 0 & 0 \\ h_2q_1 & d_2 & q_2h_3 & q_2b_3h_4 \\ b_2q_1 & g_2 & d_3 & g_3 \\ 0 & 0 & h_4q_3 & d_4 \end{bmatrix}$$

Five-diagonal twisted $(H, 1)$–qs matrices

Note that twisted $(H, 1)$–qs matrices of the interlacing patterns are always five-diagonal with a familiar staircase structure:

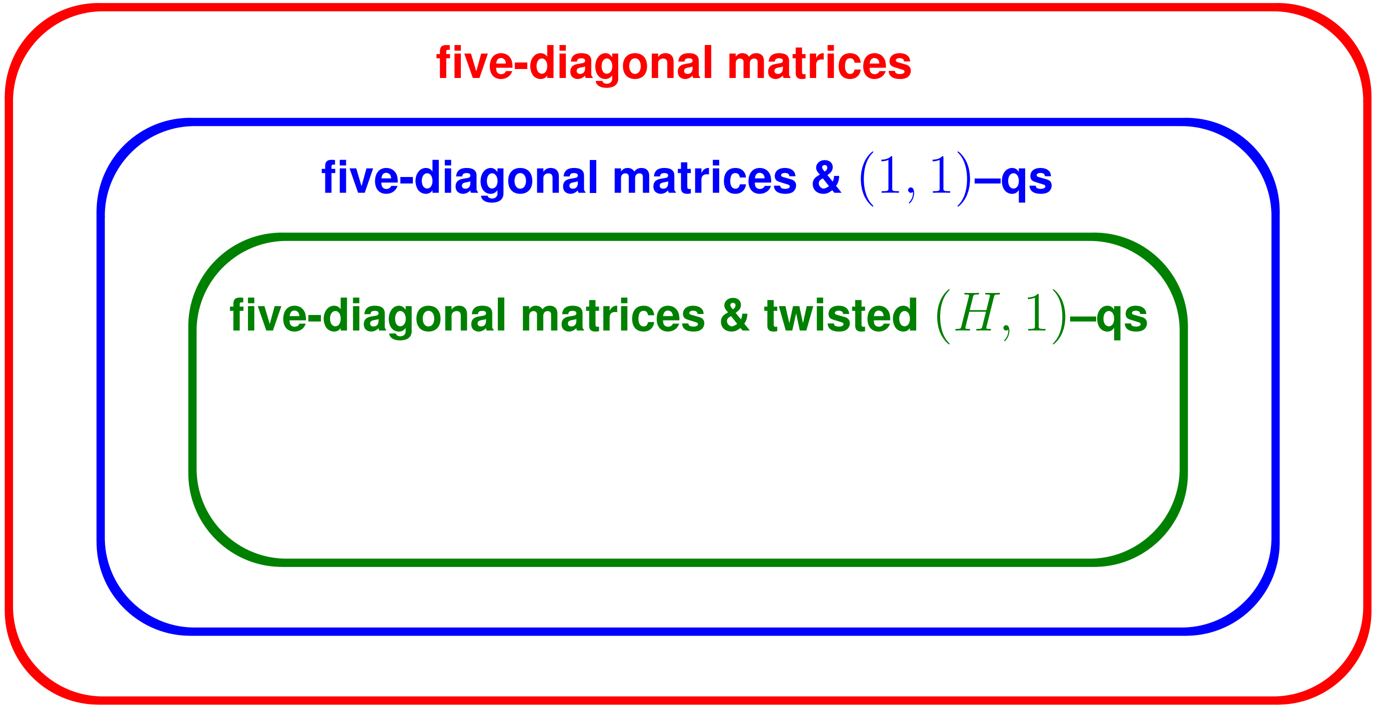
$$H(0,1,0,1,0,\dots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ q_1 h_2 & d_2 & q_2 h_3 & q_2 b_3 & & & \\ q_1 b_2 & g_2 & d_3 & g_3 & 0 & & \\ & 0 & q_3 h_4 & d_4 & q_4 h_5 & q_4 b_5 & \\ & & q_3 b_4 & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Five-diagonal twisted $(H, 1)$–qs matrices

$$H(\star, 1, 0, 1, 0, \ldots) = \begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$H(\star, 0, 1, 0, 1, \ldots) = \begin{bmatrix} \star & \star & \star & & & & & \\ \star & \star & \star & 0 & & & & \\ 0 & \star & \star & \star & \star & & & \\ & \star & \star & \star & \star & 0 & & \\ & & 0 & \star & \star & \star & \star & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Full description of five-diagonal twisted $(H, 1)$-qs matrices



General five-diagonal matrix

$$A = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

No restrictions

five-diagonal matrices & $(1,1)$–qs

$$A = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$a_{i,i+2} \cdot a_{i+1,i+3} = a_{i+2,i} \cdot a_{i+3,i+1} = 0, \quad i = 1, \ldots, n-3.$$

five-diagonal matrices & twisted $(H,1)$–qs

$$A = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \boxed{\star} & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \\ 0 & \boxed{\star} & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \boxed{a_{24}} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & \boxed{a_{42}} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$a_{i,i+2} \cdot a_{i+1,i+3} = a_{i+2,i} \cdot a_{i+3,i+1} = 0, \quad i = 1, \ldots, n-3,$$

$$a_{i,i+2} \cdot a_{i+2,i} = 0, \quad i = 1, \ldots, n-2.$$

An auxiliary result

The choice of generators of quasiseparable matrices is not unique. Hence, by taking different generators one can obtain different twisted $(H, 1)$–qs matrices

An old Fiedler matrix

$$F = \left[\begin{array}{cccccccc} -a_1 & -a_2 & 1 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & -a_3 & 0 & -a_4 & 1 & & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 0 & -a_5 & 0 & -a_6 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right]$$

Our Fiedler matrix

$$\widehat{F} = \left[\begin{array}{cccccccc} -a_1 & -a_2 & -a_3 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 1 & -a_5 & & & \\ & -a_4 & 0 & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & 1 & -a_7 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right]$$

Bijection between strongly Hessenberg matrices and polynomial systems

The following widely-known result is due to Barnett.

$$H = \begin{bmatrix} h_{0,1} & h_{0,2} & h_{0,3} & \cdots & h_{0,n} \\ h_{1,1} & h_{1,2} & h_{1,3} & \cdots & h_{1,n} \\ 0 & h_{2,2} & h_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & h_{n-2,n} \\ 0 & \cdots & 0 & h_{n-1,n-1} & h_{n-1,n} \end{bmatrix}, \quad \{\lambda_0, \lambda_n\} = \{\frac{1}{h_{0,0}}, \frac{1}{h_{n,n}}\}.$$

$$\Uparrow$$

BIJECTION

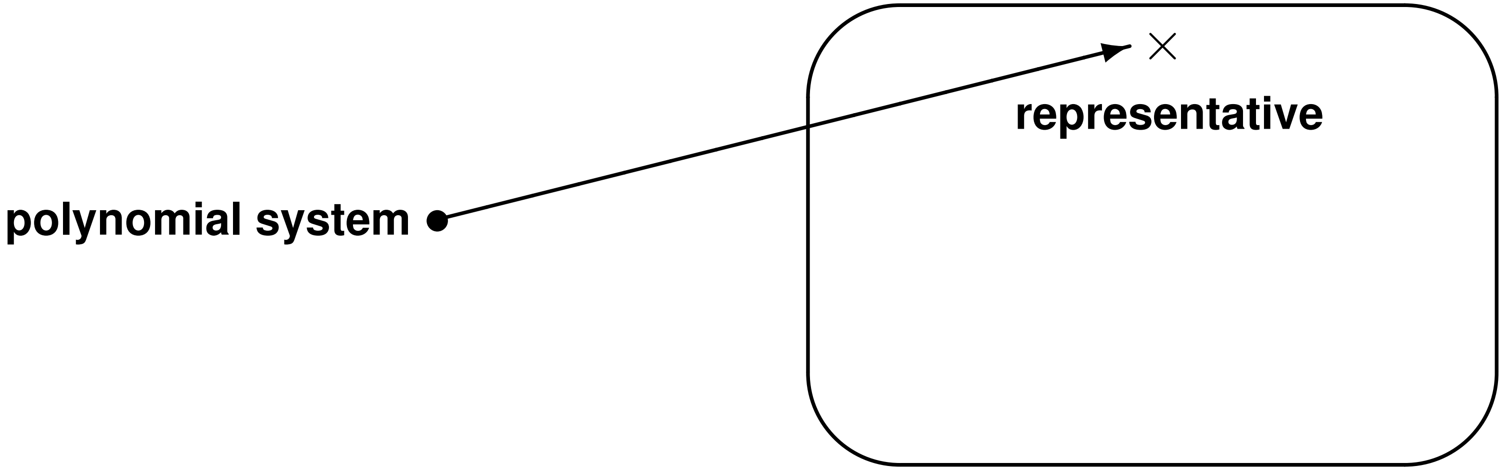
$$\Downarrow$$

$$r_0(x) = \lambda_0, \quad r_k(x) = \lambda_0 \lambda_1 \ldots \lambda_k \det(xI - H_{k\times k}), \quad k = 1, \ldots, n.$$

Correspondence between five-diagonal twisted $(H,1)$–qs matrices and polynomial systems

Of course there cannot be any bijections between polynomial systems and five-diagonal matrices because for a given system of polynomials there can exist infinitely many five-diagonal matrices related to it.

But one can try to find a good representative in the family of five-diagonal matrices related to the given system of polynomials and establish the bijection in this sense.



Case 1. General three-term recurrence relations

Theorem 1. A system of polynomials $R = \{r_k(x)\}_{k=0}^{n}$ satisfies three-term recurrence relations

$$r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x),$$
$$r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0.$$

if and only if it is related to a matrix A of the following zero pattern

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \colorbox{lightgray}{$\star$} & \star & \colorbox{lightgray}{$\star$} & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \colorbox{lightgray}{$\star$} & \star & \colorbox{lightgray}{$\star$} & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with nonzero highlighted entries.

Case 1. General three-term recurrence relations

Example 1. Fiedler matrix

$$F^T = \begin{bmatrix} -a_1 & 1 & 0 & & & & & \\ -a_2 & 0 & -a_3 & 1 & & & & \\ 1 & 0 & 0 & 0 & 0 & & & \\ & 0 & -a_4 & 0 & -a_5 & 1 & & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

If $a_k \neq 0$, then the highlighted entries in matrix F^T are not zeros and Horner polynomials satisfy general three-term recurrence relations:

$$p_k(x) = (x + \frac{a_k}{a_{k-1}})p_{k-1}(x) - \frac{a_k}{a_{k-1}} x \cdot p_{k-2}.$$

Case 1. General three-term recurrence relations

Example 2. CMV matrix

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

If $\rho_k \neq 0$, then the highlighted entries in matrix K are not zeros and Szegö polynomials satisfy general three-term recurrence relations (**Geronimus result**):

$$\phi_0^{\#}(x) = \frac{1}{\mu_0}, \quad \phi_1^{\#}(x) = \frac{1}{\mu_1}(x \cdot \phi_0^{\#}(x) + \rho_1\rho_0^* \cdot \phi_0^{\#}(x)),$$

$$\phi_k^{\#}(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^{\#}(x), \quad k = 2, \ldots, n.$$

Case 2. Szegö-type two-term recurrence relations

Theorem 2. A system of polynomials $R = \{r_k(x)\}_{k=0}^{n}$ satisfies Szegö-type two-term recurrence relations

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (x+\theta_k)\cdot r_{k-1}(x) \end{bmatrix}$$

with $\alpha_k\delta_k - \beta_k\gamma_k \neq 0$, $\delta_k \neq 0$ if and only if it is related to a matrix A of the following zero pattern

$$\begin{bmatrix} \star & \star & 0 & & & & \\ \star & \star & \star & \boxed{\star} & & & \\ \boxed{\star} & \star & \star & \star & 0 & & \\ & 0 & \star & \star & \star & \boxed{\star} & \\ & & \boxed{\star} & \star & \star & \star & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with nonzero highlighted entries.

Case 2. Szegö-type two-term recurrence relations

Example 1. Fiedler matrix

$$F^T = \begin{bmatrix} -a_1 & 1 & 0 & & & & & \\ -a_2 & 0 & -a_3 & \boxed{1} & & & & \\ \boxed{1} & 0 & 0 & 0 & 0 & & & \\ & 0 & -a_4 & 0 & -a_5 & \boxed{1} & & \\ & & \boxed{1} & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Highlighted entries in the matrix are **always** not zeros. Hence, Horner polynomials satisfy two-term Szegö-type recurrence relations without any restrictions:

$$\begin{bmatrix} F_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_k & 1 \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ x \cdot p_{k-1}(x) \end{bmatrix}.$$

Case 2. Szegö-type two-term recurrence relations

Example 2. CMV matrix

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

Highlighted entries in the matrix are **always** not zeros. Hence, Szegö polynomials satisfy two-term Szegö-type recurrence relations without any restrictions (**this is well-known**):

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}.$$

Case 3. *EGO*–type two-term recurrence relations

Theorem 3. A system of polynomials $R = \{r_k(x)\}_{k=0}^n$ satisfies *EGO*–type recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \beta_k & \gamma_k \\ \delta_k & \theta_k x + \varepsilon_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}.$$

if and only if it is related to some five-diagonal twisted $(H, 1)$–qs matrix.

Remark. Five diagonal twisted $(H, 1)$–qs matrices $A = [a_{ij}]$ are those satisfying

$$a_{i,i+2} \cdot a_{i+1,i+3} = a_{i+2,i} \cdot a_{i+3,i+1} = 0, \quad i = 1, \ldots, n-3,$$
$$a_{i,i+2} \cdot a_{i+2,1} = 0, \quad i = 1, \ldots, n-2.$$

Case 3. *EGO*–type two-term recurrence relations

Examples. Horner and Szegö polynomials

Horner polynomials satisfy the following *EGO*–type two-term recurrence relations:

$$\begin{bmatrix} F_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_1(x) \\ p_1(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x + a_1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_k & x \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ p_{k-1}(x) \end{bmatrix}.$$

Similarly, Szegö polynomials satisfy

$$\begin{bmatrix} F_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\mu_0} \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \begin{bmatrix} \mu_k & \rho_{k-1}^{*}\mu_k \\ \frac{\rho_k}{\mu_k} & \frac{1}{\mu_k}x - \frac{\rho_{k-1}^{*}\rho_k}{\mu_k} \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ \phi_{k-1}^{\#}(x) \end{bmatrix}.$$

Part 2. Twisted Green's matrices. Factorizations. Multiplication operators.

Joint work with **Vadim Olshevsky** and **Gilbert Strang**.

Factorizations

Unitary Hessenberg and CMV matrices

It is well-known that Unitary Hessenberg matrix is factorizable into the product of Givens rotations (**the so-called Schur representation**):

$$M = \Gamma_0 \Gamma_1 \Gamma_2 \dots \Gamma_n,$$

$$\Gamma_0 = \left[\begin{array}{c|c} \rho_0^* & \\ \hline & I_{n-1} \end{array}\right], \quad \Gamma_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -\rho_k & \mu_k & \\ & \mu_k & \rho_k^* & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Gamma_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -\rho_n \end{array}\right].$$

It was proved by Cantero, Moral and Velázquez that CMV matrix has similar factorization:

$$K = [\Gamma_0 \Gamma_2 \dots] \cdot [\Gamma_1 \Gamma_3 \dots]$$

Even terms first, then odd terms!

Factorizations

Companion and Fiedler matrices

Companion matrix admits similar factorization:

$$C = A_1 A_2 \dots A_n,$$

$$A_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & -a_k & 1 & \\ & 1 & 0 & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad A_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & -a_n \end{array}\right].$$

And Fiedler matrix is factorizable into the product of the same terms but in a different order:

$$F = [A_1 A_3 \dots] \cdot [A_2 A_4 \dots]$$

Odd terms first, then even terms!

General $(H,1)$-qs matrices are not factorizable

Example. Consider a 3×3 twisted $(H,1)$-qs matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Assume that it has a factorization

$$A = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & g & h \end{bmatrix} = \begin{bmatrix} a & bf & bg \\ c & df & dg \\ 0 & h & e \end{bmatrix},$$

then coefficients $\{b, d, f, g\}$ must obey a system of equations

$$\begin{cases} bg = df = 0 \\ bf = dg = 1 \end{cases}$$

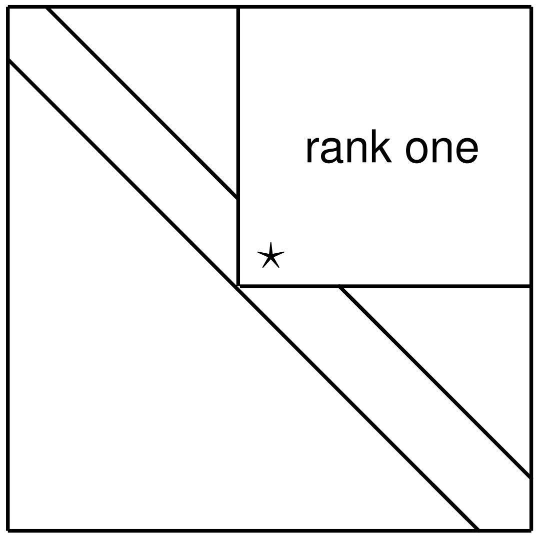
which is inconsistent. $\Longrightarrow$ **Too wide class!**

Factorizable $(H,1)$-qs matrices

Green's matrices

Definition. [Rank definition of Green's matrices] A strictly upper Hessenberg matrix G is called *Green's* $(H,1)$*-qs* if

$$\max_{1\leqslant i\leqslant n} \operatorname{rank} G(1:i, i:n) = 1.$$



Generators of Green's matrices.

Definition. [Generator definition of Green's matrices] A strictly upper Hessenberg matrix G is called *Green's* $(H, 1)$*-qs* if it can be represented in the form

$$G = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\tau}_0\sigma_1\tau_2 & \widehat{\tau}_0\sigma_1\sigma_2\tau_3 & \cdots & \cdots & \widehat{\tau}_0\sigma_1\ldots\sigma_{n-1}\tau_n \\ \widehat{\sigma}_1 & \widehat{\tau}_1\tau_2 & \widehat{\tau}_1\sigma_2\tau_3 & \cdots & \cdots & \widehat{\tau}_1\sigma_2\ldots\sigma_{n-1}\tau_n \\ 0 & \widehat{\sigma}_2 & \widehat{\tau}_2\tau_3 & \cdots & \cdots & \widehat{\tau}_2\sigma_3\ldots\sigma_{n-1}\tau_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \widehat{\sigma}_{n-2} & \widehat{\tau}_{n-2}\tau_{n-1} & \widehat{\tau}_{n-2}\sigma_{n-1}\tau_n \\ 0 & \cdots & \cdots & 0 & \widehat{\sigma}_{n-1} & \widehat{\tau}_{n-1}\tau_n \end{bmatrix},$$

where $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ are called *generators* of G.

Factorization of Green's and twisted Green's matrices.

$$G = \Theta_0 \Theta_1 \cdots \Theta_{n-1} \Theta_n,$$

$$\Theta_0 = \left[\begin{array}{c|c} \widehat{\tau}_0 & \\ \hline & I_{n-1} \end{array}\right], \quad \Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & \tau_k & \sigma_k & \\ & \widehat{\sigma}_k & \widehat{\tau}_k & \\ \hline & & & I_{n-k-1} \end{array}\right], \quad \Theta_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & \tau_n \end{array}\right].$$

Theorem. Let G be a twisted Green's matrix of pattern $(i_1, i_2, \ldots, i_n)$ with generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$. Then it can be constructed by the following procedure:

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } i_k = 0, \\ \Theta_k^T G_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots n, \quad \text{and} \quad G = G_n.$$

Corollary. All the matrices in the Theorem above have the same system of characteristic polynomials of leading submatrices.

New twist transformation.

It turns out that matrices in the factorization above can be interchanged without transposition!

Theorem. Let G be a Green's matrix of size n having generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$ and $(j_1, j_2, \ldots, j_n)$ be an arbitrary sequence of binary digits. Then all 2^n matrices $G(j_1, j_2, \ldots, j_n)$ constructed via the following procedure:

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } i_k = 0, \\ \Theta_k G_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots n, \quad G(j_1, j_2, \ldots, j_n) = G_n$$

share the same system of characteristic polynomials.

Since these matrices don't coincide in general with twisted Green's ones, this theorem introduces a new kind of twist transformation.

Five-diagonal twisted Green's matrices

Lemma. A pentadiagonal matrix A is twisted Green's of pattern $(0, 1, 0, 1, \dots)$ if and only if it has the following zero pattern:

$$\begin{bmatrix} \star & \star & 0 & & & & \\ \star & \star & \star & \star & & & \\ \star & \star & \star & \star & 0 & & \\ & 0 & \star & \star & \star & \star & \\ & & \star & \star & \star & \star & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} = \begin{bmatrix} \widehat{\tau}_0\tau_1 & \widehat{\sigma}_1\tau_2 & \widehat{\sigma}_1\sigma_2 & & & & & & \\ \widehat{\tau}_0\sigma_1 & \widehat{\tau}_1\tau_2 & \widehat{\tau}_1\sigma_2 & 0 & & & & & \\ 0 & \widehat{\sigma}_2\tau_3 & \widehat{\tau}_2\tau_3 & \widehat{\sigma}_3\tau_4 & \widehat{\sigma}_3\sigma_4 & & & & \\ & \widehat{\sigma}_2\sigma_3 & \widehat{\tau}_2\sigma_3 & \widehat{\tau}_3\tau_4 & \widehat{\tau}_3\sigma_4 & 0 & & & \\ & & 0 & \widehat{\sigma}_4\tau_5 & \widehat{\tau}_4\tau_5 & \widehat{\sigma}_5\tau_6 & \widehat{\sigma}_5\sigma_6 & & \\ & & & \widehat{\sigma}_4\sigma_5 & \widehat{\tau}_4\sigma_5 & \widehat{\tau}_5\tau_6 & \widehat{\tau}_5\sigma_6 & 0 & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with rank-one 2×2 marked blocks.

Recurrence relations for Green's polynomials

Theorem. [Recurrence relations for Green's polynomials] Let G be an $n \times n$ Green's matrix having generators $\{\sigma_k, \tau_k, \widehat{\sigma}_k, \widehat{\tau}_k\}$, then a system of polynomials $\{r_k(x)\}_{k=0}^{n}$ is related to it via if and only if polynomials $r_k(x)$ satisfy two-term recurrence relations

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix},$$

Remark. Recurrence relations via generators

$$\begin{bmatrix} f_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{\widehat{\sigma}_k} \begin{bmatrix} \widehat{\sigma}_k \sigma_k - \widehat{\tau}_k \tau_k & \widehat{\tau}_k \\ -\tau_k & 1 \end{bmatrix} \begin{bmatrix} f_{k-1}(x) \\ x \cdot r_{k-1}(x) \end{bmatrix},$$

Multiplication operators

Let M be an infinite-dimensional unitary Hessenberg matrix and $\{\phi_k^{\#}(x)\}_{k\geqslant 0}$ be the infinite sequence of polynomials orthogonal on the unit circle related to M, then

$$\big[\phi_0^{\#}(x)\ \phi_1^{\#}(x)\ \phi_2^{\#}(x)\cdots\big]M = x\big[\phi_0^{\#}(x)\ \phi_1^{\#}(x)\ \phi_2^{\#}(x)\cdots\big].$$

If M_n is of size n and λ is a root of polynomial $\phi_n^{\#}(x)$, then

$$\big[\phi_0^{\#}(\lambda)\ \phi_1^{\#}(\lambda)\cdots\phi_{n-1}^{\#}(\lambda)\big]M_n = \lambda\big[\phi_0^{\#}(\lambda)\ \phi_1^{\#}(\lambda)\cdots\phi_{n-1}^{\#}(\lambda)\big]$$

i.e. vector $\big[\phi_0^{\#}(\lambda)\ \phi_1^{\#}(\lambda)\cdots\phi_{n-1}^{\#}(\lambda)\big]$ is the **left eigenvector** of M_n corresponding to the **eigenvalue** λ.

Define for Szegö polynomials $\{\phi_k^{\#}(x)\}_{k\geqslant 0}$ right **Laurent polynomials** as follows:

$$\chi_k(x) = \begin{cases} x^{-l}\phi_k(x) & k = 2l, \\ x^{-l}\phi_k^{\#}(x) & k = 2l+1, \end{cases}$$

It turns out that an infinite-dimensional CMV matrix K play the same role for **Laurent polynomials** $\{\chi_k(x)\}_{k\geqslant 0}$ as unitary Hessenberg does for Szegö polynomials i.e.

$$\big[\chi_0(x)\ \chi_1(x)\ \chi_2(x)\cdots\big]K = x\big[\chi_0(x)\ \chi_1(x)\ \chi_2(x)\cdots\big].$$

$$\big[\chi_0(\lambda)\ \chi_1(\lambda)\cdots\chi_{n-1}(\lambda)\big]K_n = \lambda\big[\chi_0(\lambda)\ \chi_1(\lambda)\cdots\chi_{n-1}(\lambda)\big].$$

Twisted Green's matrices as multiplication operators

Let $\mathcal{J} = (j_1, j_2, j_3, \dots)$ be an infinite sequence of binary digits. We define twisted Green's matrices $G_{\mathcal{J}}$ by the recursion

$$G_0 = \Theta_0, \quad G_k = \begin{cases} G_{k-1}\Theta_k & \text{if } j_k = 0, \\ \Theta_k G_{k-1} & \text{if } j_k = 1, \end{cases}, \quad G_{\mathcal{J}} = G_\infty,$$

Matrices $G_{\mathcal{J}}$ are related to the same polynomials as G For every $\mathcal{J}$ we also define a sequence of Laurent polynomials $\{\psi_k(x)\}_{k\geqslant 0}$:

$$\psi_k(x) = \begin{cases} x^{-\sum_{m=1}^{k+1} j_m} r_k(x) & \text{if } j_{k+1} = 0, \\ x^{-\sum_{m=1}^{k+1} j_m} f_k(x) & \text{if } j_{k+1} = 1, \end{cases}$$

Theorem.

(i) $\left[\psi_0(x)\ \psi_1(x)\ \psi_2(x) \cdots\right] G_{\mathcal{J}} = x\left[\psi_0(x)\ \psi_1(x)\ \psi_2(x) \cdots\right].$

(ii) If $\lambda \neq 0$, then $\left[\psi_0(\lambda)\ \psi_1(\lambda) \cdots \psi_{n-1}(\lambda)\right] G_{\mathcal{J}_n} = \lambda\left[\psi_0(\lambda)\ \psi_1(\lambda) \cdots \psi_{n-1}(\lambda)\right].$

Thank you for your attention!