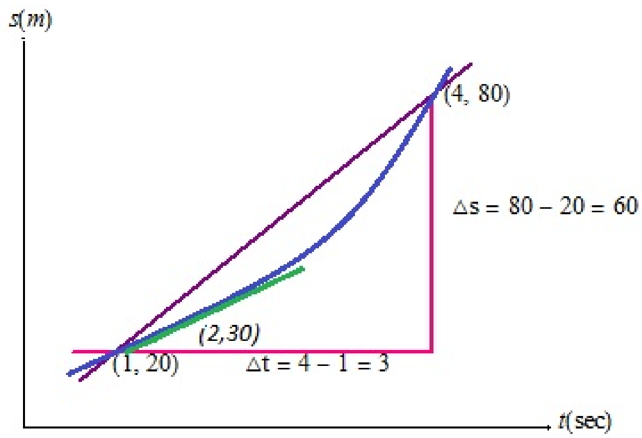
Chapter 2: *The Derivative*

Lesson 2.1: *Velocity*

Suppose a car is accelerating down the highway and that the graph of its displacement s as a function of time t is as follows:



Suppose we want to know how fast the car is traveling at $t = 1$ second. One attempt to do this is to calculate average velocity between $t = 1$ second and $t = 4$ seconds.

$$\frac{\triangle s}{\triangle t} = \frac{s(4) - s(1)}{4 - 1} = \frac{80 - 20}{3} = 603 = 20m/s$$

The problem with this is that as the car is accelerating, this will be higher than the 'actual' speed at $t = 1$ second.
A better approximation would to be take the average velocity between $t = 1$ second and $t = 2$ seconds as the velocity has less time to change over a shorter time interval.

$$\frac{\triangle s}{\triangle t} = \frac{s(2) - s(1)}{2 - 1} = \frac{30 - 20}{1} = 101 = 10m/s$$

Of course, using high speed photography, we could do better if we measured over time intervals of $\frac{1}{2}$ second. In the limit we would get

$$\lim_{h \to 0} \frac{s(1 + h) - s(1)}{h}$$

and we would then call this the actual or instantaneous velocity at $t = 1$ second.
Definition:
For an object whose displacement function as a function of time t is $s(t)$, we define the instantaneous velocity at time $t = a$, $v(a)$ by

$$v(x) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

Example:

If $s(t) = t^2$, find $v(3)$.

$$v(3) = \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} 6 + h \qquad \text{cancelation okay since } h \neq 0 \text{ for a limit at } 0$$

$$= \lim_{h \to 0} 6 + \lim_{h \to 0} h \qquad \text{Law 2}$$

$$= 6 + 0 \qquad \text{Law 5, 6}$$

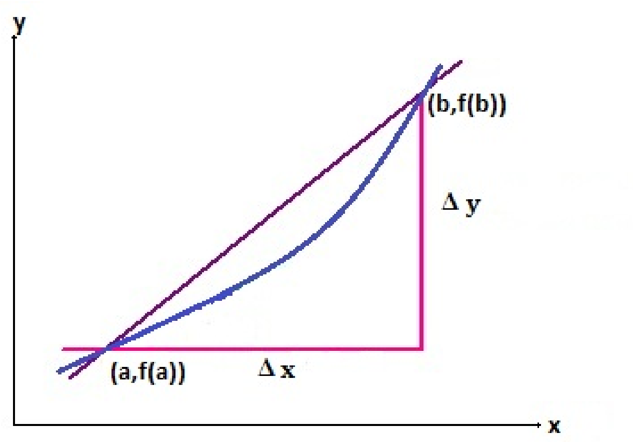
$$= 6$$

The Derivative:

Average Rate of Change:

For a function f, we define the average rate of change of f over an interval $[a,b]$ to be

$$\frac{\triangle y}{\triangle x}=\frac{f(b)-f(a)}{b-a}$$



This is the slope of the line passing through $(a,f(a)),(b,f(b))$.

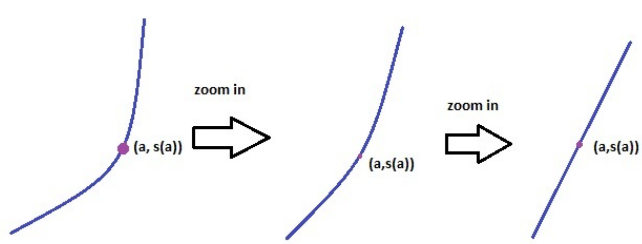
Back to the car example...

When we found our first average velocity between $t=1$ second and $t=4$ seconds, from the graph we say that this was actually the slope of the line joining the points $(1,20)$ and $(4,80)$.

In general, the average velocity over a time interval $a\le t\le b$ is the slope of the line joining the points $(a,s(a))$ and $(b,s(b))$ on the graph of $s(t)$ which corresponds to $t=a$ and $t=b$ respectively.

When we take limits by considering smaller and smaller time intervals, we get the instantaneous velocity $v(a)$, which is the slope of the graph of $s(t)$ at the point $a,s(a))$ corresponding to $t=a$.

In pictures,



Taking shorter time intervals is like zooming in, and the more we zoom in, the more the graph near $(a,s(a))$ looks like a straight line.

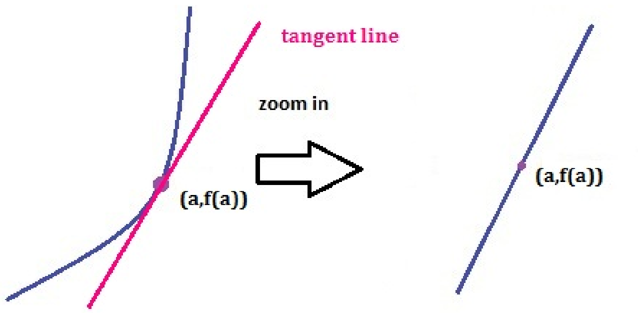
Instantaneous Rate of Change - Derivative:

The derivative (or instantaneous rate of change) of f at a, $f'(a)$, is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$$

provided this limit exists. If the limit does exist, we say that f is differentiable at $x = a$.
Sometimes we write $\frac{df}{dx}\mid_{x=0}$ instead of $f'(a)$.
In terms of slope the derivative can be interpreted as:

- the slope of the tangent line to the curve at $(a, f(a))$
- the slope of the curve at $(a, f(a))$



If f is differentiable at $x = a$, then the graph of f near $(a, f(a))$ looks like a straight line. The equation of the tangent line to the graph at $(a, f(a))$ is

$$(y - f(a)) = f'(a)(x - a)$$

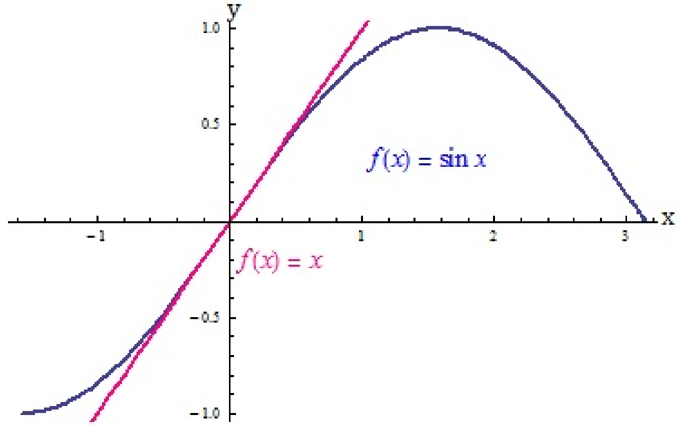
or

$$y = f(a) + f'(a)(x - a)$$

This tangent line give us a linear approximation of the function near $x = a$.

Example:

Looking at the graph near $(0, 0)$, we see that the derivative of $\sin x$ appears to be 1.



Hence, $(y - 0) = 1(x - 0)$, ie $y = x$, gives us a linear approximation to $\sin x$ for x small, (near 0).
e.g $\sin 0.1 = 0.0998 \approx 0.1$

Example:

Find the equation of the tangent line to the graph of $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

The procedure for these problems is always the same. First we need to find the slope, ie $f'(2)$. By definition:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2-(2+h)}{(2+h)(2)}}{h} && \text{Common Denominator} \\
&= \lim_{h\to 0} \frac{\frac{-h}{(2+h)(2)}}{h} \\
&= \lim_{h\to 0} \frac{-1}{(2+h)(2)} && h \neq 0 \text{ for limit at } 0 \\
&= -\lim_{h\to 0} \frac{1}{(2+h)(2)} \\
&= -\lim_{h\to 0} \frac{1)}{2+h} \lim_{h\to 0} \cdot \frac{1}{2} \\
&= -\frac{1}{2} \cdot \frac{1}{2} \\
&= -\frac{1}{4}
\end{aligned}$$

Equation of the tangent line is $y - f(a) = f'(a)(x - a)$.
Here, $x = 2, f(a) = \frac{1}{2}, f'(a) = -\frac{1}{4}$, so

$$(y - \frac{1}{2}) = -\frac{1}{4}(x - 2)$$

$$y - \frac{1}{2} = -\frac{x}{4} + \frac{1}{2}$$

$$y = -\frac{x}{4} + 1$$