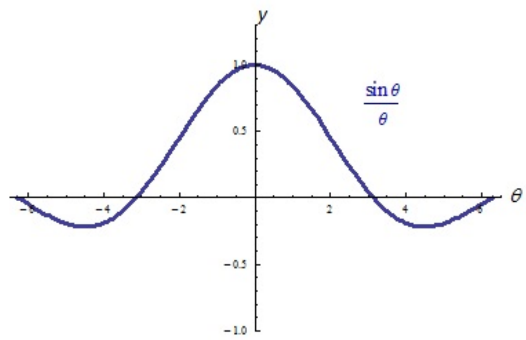
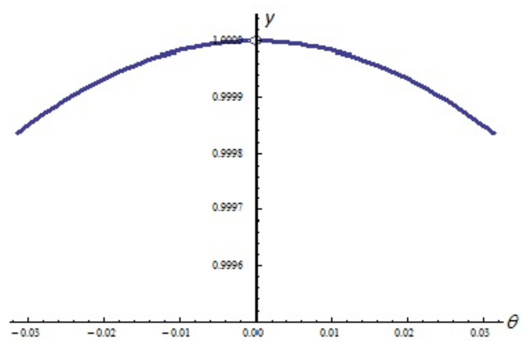
LESSON 1.8: *Limits*

Consider the graph of $\frac{\sin\theta}{\theta}$, ($\theta$ in radians), for θ near 0 (but not actually 0).



The value of $\frac{\sin\theta}{\theta}$ seems to get close to 1 as θ gets close to zero. By zooming in on the graph around the point $(0,1)$



it seems that we can make the value of $\frac{\sin\theta}{\theta}$ as close to 1 as we like, provided we take θ sufficiently close to 0.

This is the idea behind a limit.

A function f is defined on an interval c, except possibly at the point $x = c$. We define the limit of the function $f(x)$ as x approaches c, written $\lim_{x\to c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write

$$\lim_{x\to c} f(x) = L$$

A very useful result for breaking complicated limits down into simpler ones is Theorem 1.2.

Theorem 1.2: Laws of Limits

1: If b is constant, then $\lim_{x\to c}(bf(x)) = b\lim_{x\to c}(f(x))$.

2: $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c}(f(x)) \pm \lim_{x\to c}(g(x))$

3: $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c}(f(x)))(\lim_{x\to c}(g(x)))$

4: $\lim_{x\to c}(\frac{f(x)}{g(x)}) = \frac{\lim_{x\to c}(f(x))}{\lim_{x\to c}(g(x))}$

5: For any constant k, $\lim_{x\to c}(k) = k$

6: $\lim_{x\to c}(x) = c$

Examples:

1.

$$\lim_{x\to 1}(2x+3) = \lim_{x\to 1}(2x) + \lim_{x\to 1} 3 \qquad \text{Law 2}$$

$$= 2\lim_{x\to 1}(x) + \lim_{x\to 1} 3 \qquad \text{Law 1}$$

$$= 2\cdot 1 + 3 \qquad \text{Law 5,6}$$

$$= 5$$

2.

$$\lim_{x\to 3}\frac{x^2+5x}{x+9} = \frac{\lim_{x\to 3} x^2+5x}{\lim_{x\to 3} x+9} \qquad \text{Law 4}$$

$$= \frac{\lim_{x\to 3} x^2 + \lim_{x\to 3} 5x}{\lim_{x\to 3} x + \lim_{x\to 3} 9} \qquad \text{Law 2}$$

$$= \frac{(\lim_{x\to 3} x)^2 + 5\lim_{x\to 3} x}{\lim_{x\to 3} x + \lim_{x\to 3} 9} \qquad \text{Laws 1,3}$$

$$= \frac{3^2+5\cdot 3}{3+9} \qquad \text{Laws 5,6}$$

$$= 2$$

3.

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

We have to be careful here as we cannot just use Law 4 to take a quotient of limits here as $\lim_{x \to 3} x- = 0$ (otherwise we get $\frac{0}{0}$ which makes no sense).

In situations like this, one resorts to algebraic tricks.

A very common one is the difference of two square, $a^2 - b^2 = (a - b)(a + b)$.

Here

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3}$$

$$= \lim_{x \to 3} (x + 3)$$

$$= \lim_{x \to 3} x + \lim_{x \to 3} 3 \qquad \text{Law 2}$$

$$= 3 + 3 \qquad \text{Law 5,6}$$

$$= 6$$

4.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$$

Same $\frac{0}{0}$ problem as before. Trick to dealing with this is to multiply above and below by the conjugate expression $\sqrt{x}+1$ (which is okay near 1) and then use the difference of two squares, $x = 1$.

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = \lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x}+1)}{x-1} \qquad \text{by difference of two squares}$$

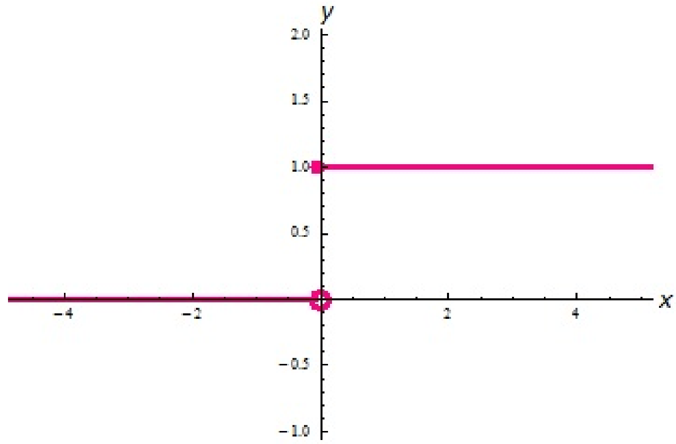
$$= \lim_{x\to 1} \sqrt{x}+1 \qquad \text{cancellation okay as } x \neq 1 \text{ for limit at } 1$$

$$= \sqrt{\lim_{x\to 1} x}+1 \qquad \text{Law 2}$$

$$= \sqrt{1}+1$$

$$= 2$$

Consider again the Heaviside function, $H(x)$.



If we approach x from the left, $H(x)$ is 0 and if we approach from the right, $H(x)$ is 1. Say $H(x)$ has limit 0 from the left and limit 1 from the write, and we write

$$\lim_{x \to 0_-} H(x) = 0, \lim_{x \to 0_+} H(x) = 1$$

Formal definition for left- and right-handed limits are similar to that for a two-sided limit.

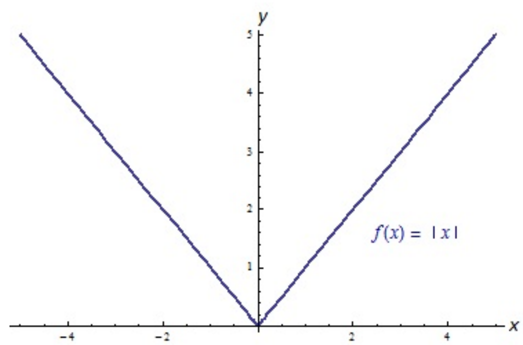
When Limits do not exist:

Example 5:

Explain why $\lim_{x\to 2} \frac{|\ x-2\ |}{x-2}$ does not exist.

Recall

$$\begin{aligned} |\ x\ | &= x, x \geq 0 \\ &- x, x < 0 \end{aligned}$$



If $x > 2$, $x - 2 > 0$, so $|\ x-2\ |= x-2$ and

$$\lim_{x\to 2_+} \frac{|\ x-2\ |}{x-2} = \lim_{x\to 2_+} x - 2x - 2 = \lim_{x\to 2_+} 1 = 1$$

(cancelation is okay since $x \neq 2$ for a limit at 2).

On the other hand, if $x < 2$, $x - 2 < 0$, so $|\ x-2\ |= -(x-2)$ and

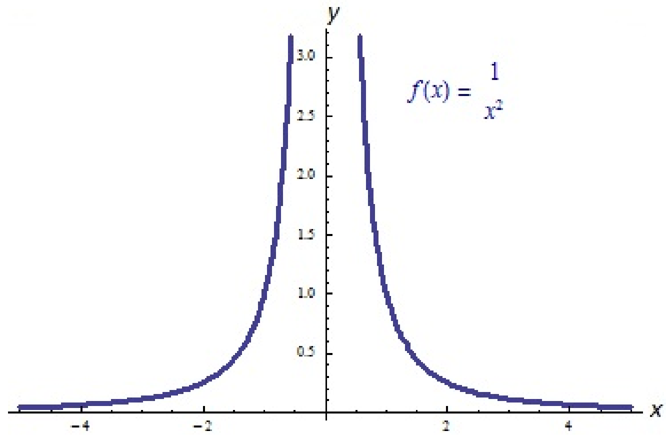
$$\lim_{x\to 2_-} \frac{|\ x-2\ |}{x-2} = \lim_{x\to 2_-} -(x-2)x - 2 = \lim_{x\to 2_-} -1 = -1$$

Since the left- and right-hand limits don't agree, the limit at 2 does not exist.

Example 6:

Explain why $\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

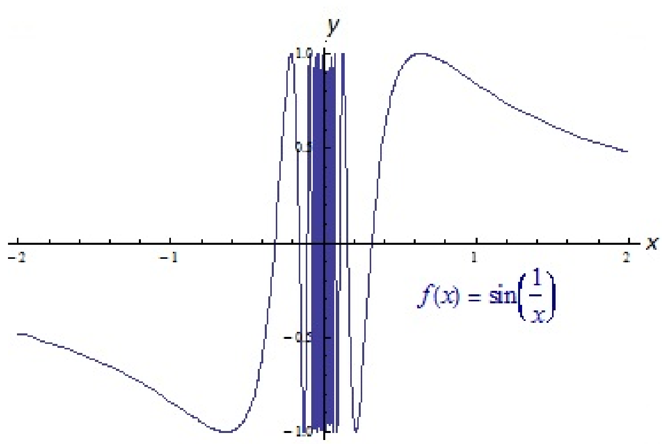
As x approaches 0, $\frac{1}{x^2}$ becomes arbitrarily large, so it cannot approach any finite number, L. Hence there can be no limit at 0.



Example 7:

Explain why $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

The sine function varies between $+1$ and -1. From the graph, we see that the function oscillates more and more as x approaches 0. There are values of x approaching 0 where $\sin(\frac{1}{x})) = 1$, and there are some where $\sin(\frac{1}{x}) = -1$. Hence, there can be no limit at 0.



Limits of Infinity:

Sometimes we want to know what happens to $f(x)$ as x gets large, i.e. the end behavior of f.

If $f(x)$ gets as close to a number L as we please when x gets sufficiently large and positive we write

$$\lim_{x\to\infty} f(x) = L.$$

If $f(x)$ gets as close to a number L as we please when x gets sufficiently large and negative we write

$$\lim_{x\to-\infty} f(x) = L.$$

Note: If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote for f.

Example 8:

$$f(x) = \frac{1}{x}$$

As x gets large in either the positive or negative directions, $\frac{1}{x}$ gets close to 0. Hence $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

Definition of Continuity:
The function, f, is continuous at c if it defined at $x = c$ and if $\lim_{x \to c} f(x) = f(c)$.

In other words, $f(x)$ is as close to $f(c)$ as we want, provided x is close enough to c. The function is continuous on an interval $[a, b]$ if it is continuous at each point of the interval (where we use one-sided limits as appropriate and the end points).

Note: For a function, f, to be continuous at c we need three things

1. f is defined at c.

2. f has a limit at c.

3. The value of this limit is $f(c)$.

If one or more of these things fail, then f is not continuous at c.

A useful result:

Theorem 1.3: Continuity of Sums, Products and Quotients of Functions:

Suppose that f and g are continuous on an interval and that b is a constant. Then, on the same interval:

1: $bf(x)$ is continuous.

2: $f(x) \pm g(x)$ is continuous.

3: $f(x)g(x)$ is continuous.

4: $\frac{f(x)}{g(x)}$ is continuous, provided $g(x)$ does not vanish on this interval.