Lesson 1.6: *Powers, Polynomials and Rational Functions*

A power function has the form

$$f(x) = kx^p$$

where k and p are constants.

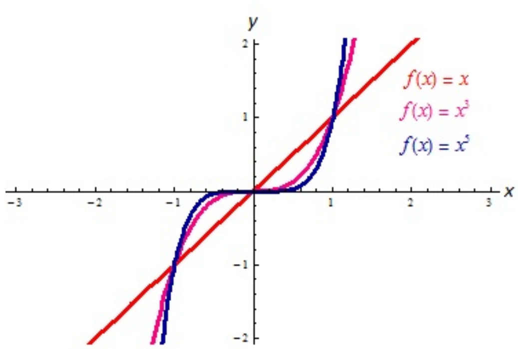
Example:

1. $V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.

2. $F = \frac{G_0 M m}{r^2}$ is Newton's Law of Gravitation.

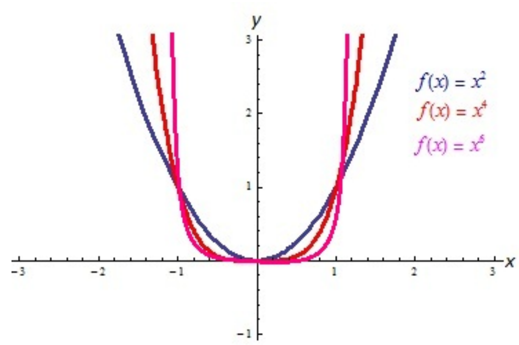
Powers of x^n:

Odd Powers



seat-shaped

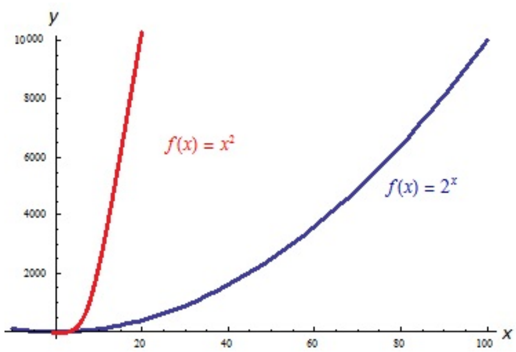
Even Powers



u-shaped

Note as we increase the power n of x, the function grows faster for x large. However, no matter how large n is, x^n still grows more slowly than any increasing exponential function. (ie Even if n is very large and a is only just about 1, for x large, $x^n \ll a^x$.)

Example:



Try seeing other examples of this using Maple or a graphing calculator.

POLYNOMIALS:

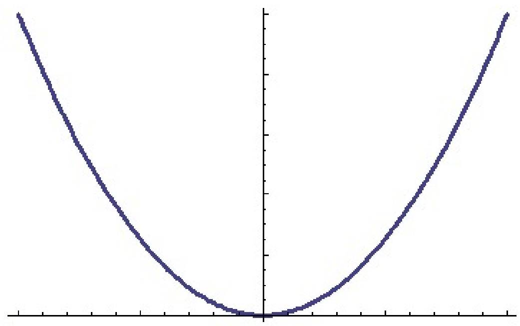
Polynomials are sums of power functions with non-negative integer exponents

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

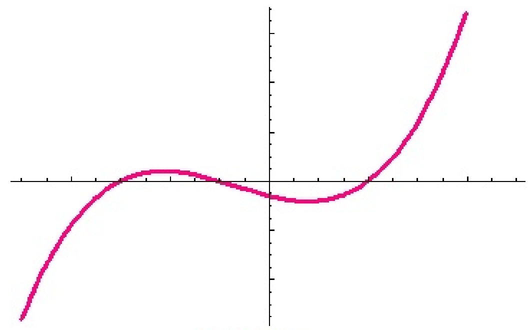
Here (provided $a_n \neq 0$), n is called the degree of the polynomial and the numbers $a_n, \ldots, a_0$ are its coefficients.

Recognizing Polynomials:

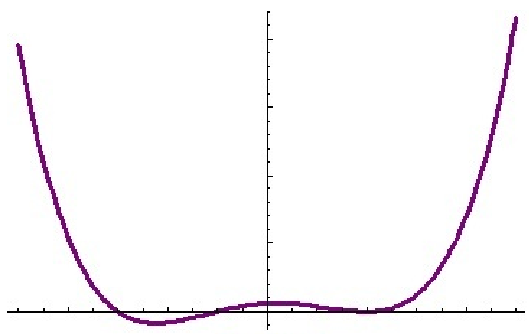
Some typical graphs:



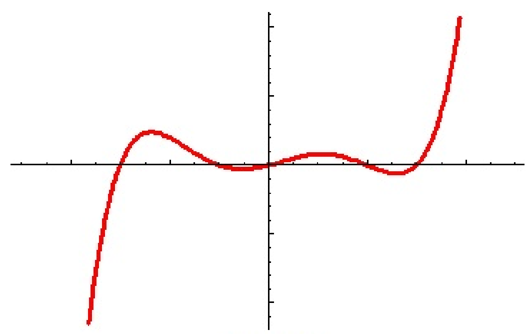
quadratic, n=2



cubic, n=3



Quartic, n=4

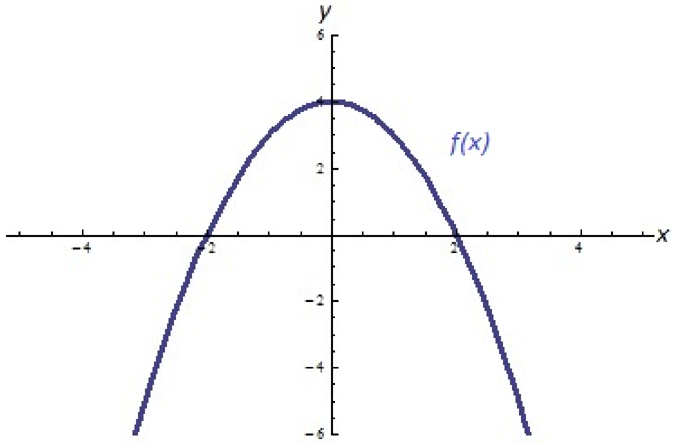


Quintic, n=5

In general, a polynomial of degree n "turns around" at most n times (and can turn around $< n$ times).

Example 1:
Find possible formulas for the polynomials with the followings graphs:

a.



First Method:

Seems to be an upside-down parabola shifted vertically upwards by 4. So a guess is $f(x) = -x^2 + 4$. This works (since $f(x) = 0 for x = -2, 2$).

Second Method (better method):

Polynomial has value 0 at $x = -2, 2$ (say $x = -2, 2$ are zeroes of $f(x)$). This means that $(x + 2)$ and $(x - 2)$ are factors of f. Hence, $f(x) = k(x + 2)(x - 2)$ for some k. To find k, we will use the y-intercept. From the graph, we see that $f(0) = 4$, so:

$$4 = k(2)(-2)$$

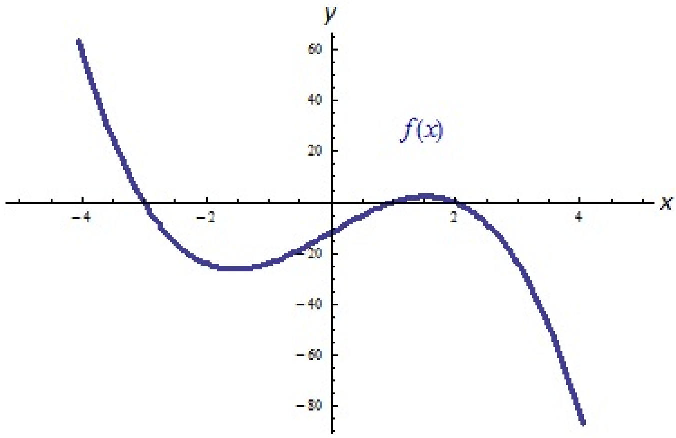
$$= -4k$$

$$k = -1$$

Hence$f(x) = -(x + 2)(x - 2) = -x^2 + 4$ again.

Example 1:

b.



This graph looks like a cubic function with zeroes at $x = -3, 1, 2$, and therefore factors of $f(x)$ are $(x+3), (x-1), (x-2)$. Thus try, $f(x) = k(x+3)(x-1)(x-2)$. Since the y-intercept is -12

$$-12 = k(3)(-1)(-2)$$

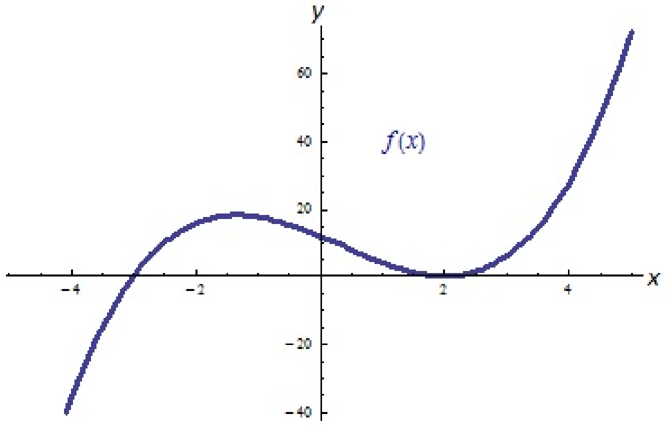
$$= 6k$$

$$k = -2$$

Hence $f(x) = -2(x+3)(x-1)(x-2)$.

Example 1:

c.



Again this looks like a cubic. Here the roots are at $x = -3$ and $x = 2$. However $x = 2$ is a double root (**Explain!!**)

Hence, $(x+3)$ and $(x-2)^2$ are factors and $f(x) = k(x+3)(x-2)^2$.

Since we have no values for f which aren't zeroes, we cannot find a specific value of k. However, the graph suggests that f get large and positive for large positive x, and large and negative for large negative x. Any positive value of k is consistent with this behavior.
(e.g. $k = 1$, $f(x) = (x+3)(x-2)^2$).

Rational Functions:

Rational functions are quotients of two polynomials:

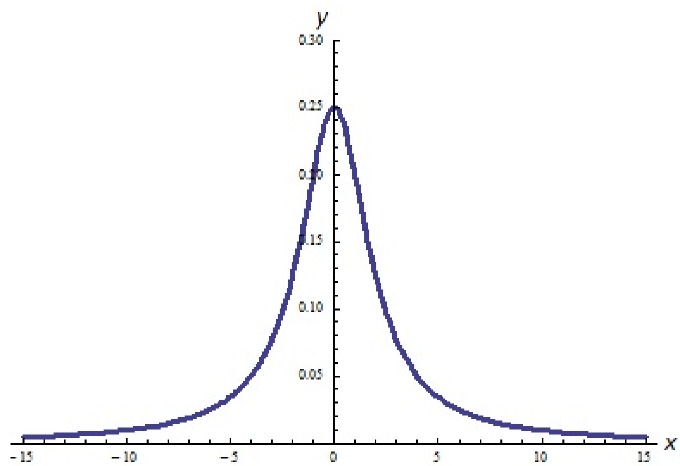
$$r(x) = \frac{P(x)}{Q(x)}$$

(usually, we assume P and Q have no common factors).

Example 2:

Look at a graph and explain the behavior of

$$y = \frac{1}{x^2+4}$$



The function is even, so the graph is symmetric about the y-axis. As x get very large (negative or positive), $\frac{1}{x^2+4}$ gets very small. Thus the graph gets arbitrarily close to the x-axis, as x increases without bound (in both directions).

Say that $y = 0$ is a horizontal asymptote of this function. We say that y tends to zero as x tends to infinity or minus infinity and this can be written as

$$y \to 0 \text{ as } x \to \infty$$

$$y \to 0 \text{ as } x \to -\infty.$$

If the graph of $y = f(x)$ approaches a horizontal line $y = L$, as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is called a horizontal asymptote. This occurs when

$$f(x) \to L \text{ as } x \to \infty,$$

$$f(x) \to L \text{ as } x \to -\infty$$

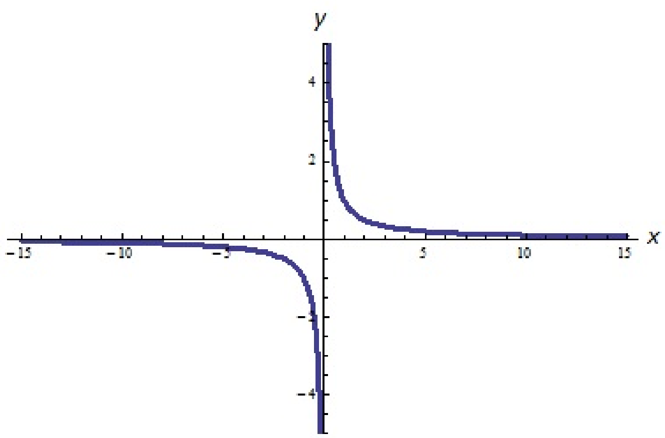
or both.

If the graph of $y = f(x)$ approaches the vertical line $x = k$ as $x \to k$ from one side or the other, that is, if $y \to \infty$ or $y \to -\infty$ when $x \to k$ then the line $x = k$ is called a vertical asymptote.

e.g:

$x = \frac{\pi}{2}$ was a vertical asymptote for $\tan x$.

$x = 0$ is a vertical asymptote for $y = \frac{1}{x}$.



Example 4:

Look at a graph and explain the behavior of

$$y = \frac{3x^2 - 12}{x^2 - 1}$$

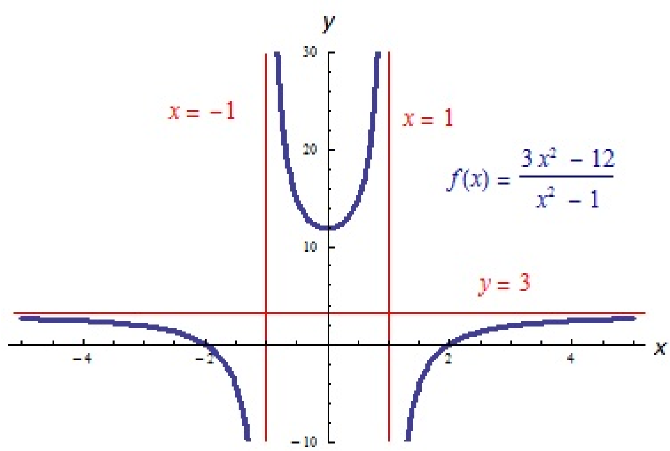
including the end behavior.

Factoring gives us

$$y = \frac{3x^2 - 12}{x^2 - 1} = \frac{3(x+2)(x-2)}{(x+1)(x-1)}.$$

From this we see that $x = -2, 2$ are zeroes (x-intercepts), while $x = -1, 1$ are vertical asymptotes. When $x \to \pm\infty$, $y = \frac{3x^2-12}{x^2-1} \approx \frac{3x^2}{x^2} = 3$ for x large.

Hence $y = 3$ is a horizontal asymptote.



Note: The function is even and so the graph is symmetric about the y-axis.