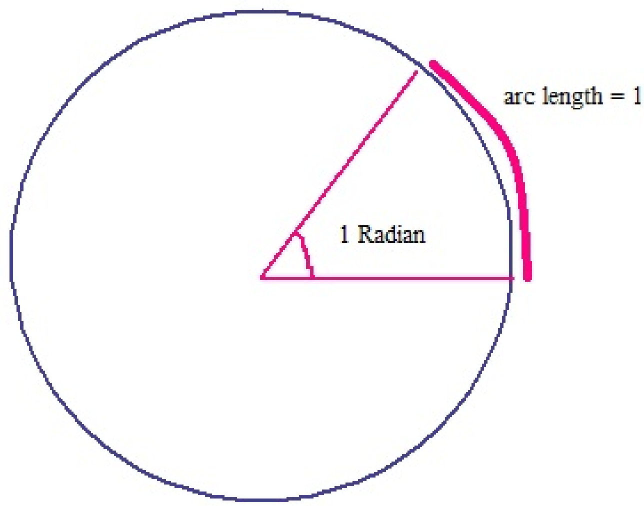
RADIANS:

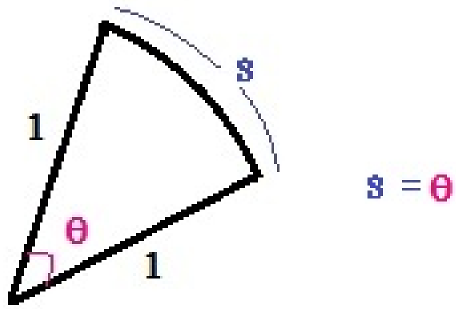
An angle of 1 radian is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. A unit circle has radius 1.



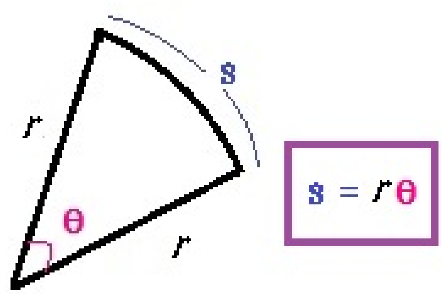
Since a circle of radius 1 has circumference $2\pi \cdot 1 = 2\pi$, we see that $360° = 2\pi$ radians.

Similarly, $180° = \pi$ radians, $60° = \frac{\pi}{3}$ radians, ect.

In general, we see that the arc length is proportional to the angle and for a circle of radius 1, the constant of proportionality being 1.



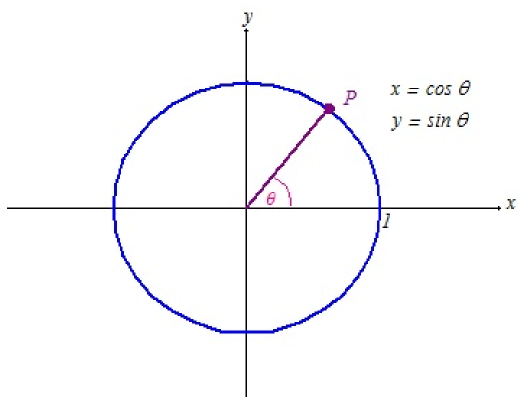
For a general circle of radius r, the arc-length is also proportional to the radius and so:



Sine and Cosine:

What I learned in secondary school...

The coordinates of the point, p, where the terminal arm of the angle, θ, intersects the circumference of the unit circle in the standard position, $(\cos(\theta), \sin(\theta))$, where θ is measured in an anti-clockwise sense starting from the positive direction x-axis.



Since the equation for the unit circle is

$$x^2 + y^2 = 1,$$

we automatically have the identity

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

As θ increases, p moves around the circle, and when θ has increased by 2π, we are back where we started. Hence $\sin(\theta)$ and $\cos(\theta)$ go through a complete cycle in 2π (and not in any smaller amount) and we say that these functions are periodic with period, 2π.

Write this as:

$$\sin(\theta + 2\pi) = \sin(\theta)$$
$$\cos(\theta + 2\pi) = \cos(\theta)$$

If $k \in \mathbb{Z}$ is any integer, then:

$$\sin(\theta + 2k\pi) = \sin(\theta)$$
$$\cos(\theta + 2k\pi) = \cos(\theta)$$

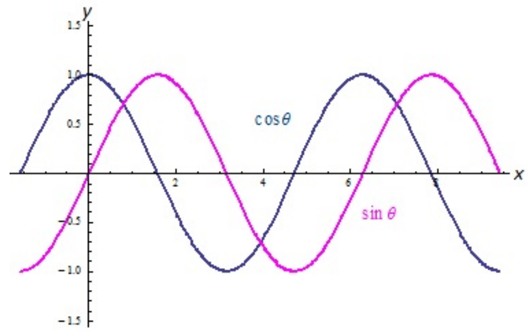
Note also:

As $\cos(\theta), \sin(\theta)$ are coordinates of a point on the unit circle, we must have that:

$$\begin{array}{c|l} -1 \leq \sin(\theta) \leq 1 & \sin(\frac{\pi}{2}) = 1, \sin(\frac{3\pi}{2}) = -1 \\ -1 \leq \cos(\theta) \leq 1 & \cos(0) = 1, \sin(\pi) = -1 \end{array}$$

for **any** value of θ.

Say $\sin(\theta), \cos(\theta)$ have amplitude 1, $(1 = \frac{1}{2}(1 - (-1)))$, half the distance between the maximum (1) and minimum (-1) values.



<u>Note:</u>

The two graphs have the same shape but the graph for $\sin(\theta)$ is that for $\cos(\theta)$ shifted $\frac{\pi}{2}$ to the right.

Hence

$$\sin(\theta) = \cos(\theta - \frac{\pi}{2})$$

Or similarly, if we replace θ with $\theta + \frac{\pi}{2}$, then

$$\sin(\theta + \frac{\pi}{2}) = \cos(\theta)$$

In general, for functions of the form

$$f(\theta) = A\sin(B\theta) + C, g(\theta) = A\cos(B\theta) + C$$

the period is $\frac{2\pi}{|B|}, B \neq 0$, and the amplitude is $\mid A \mid$.

<u>Example:</u>

1.

$$y = 5\sin(2t)$$

has period $\frac{2\pi}{|2|} = \pi$ and amplitude 5.

2.

$$y = -3\cos(\frac{t}{4})$$

has period $\frac{2\pi}{\frac{1}{4}} = 8\pi$ and amplitude $\mid -3 \mid = 3$.

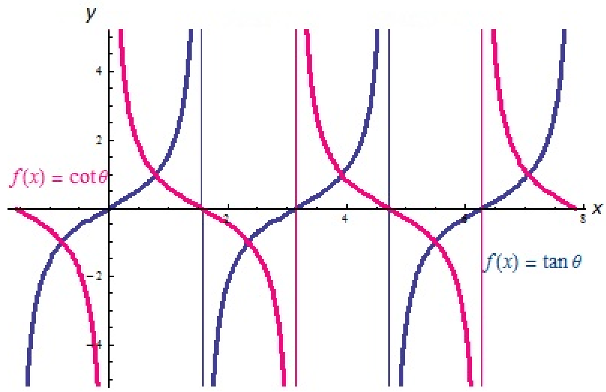
THE TANGENT AND COTANGENT FUNCTIONS:

If θ is such that $\cos(\theta) \neq 0$, (ie $\theta \notin \{\dots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\}$), then we define the tangent of θ, $\tan(\theta)$, by

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Similarly, if $\sin(\theta) \neq 0$, (ie $\theta \notin \{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}$), then we define the cotangent of θ, $\cot(\theta)$, by

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$



$\tan(\theta)$ and $\cot(\theta)$ both have period π. However, they do not have an amplitude as such since they become infinitely large at the vertical asymptotes.

If we try to solve the equation

$$\sin x = 0$$

we get infinitely many solutions

$$x \in \{\dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots\}$$

Hence $\sin x$ does not have an inverse (by horizontal line test). However, there is only one solution between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and if we pick out this solution, we do get a function, called inverse sine, $\arcsin$, or $\sin^{-1}$ defined by

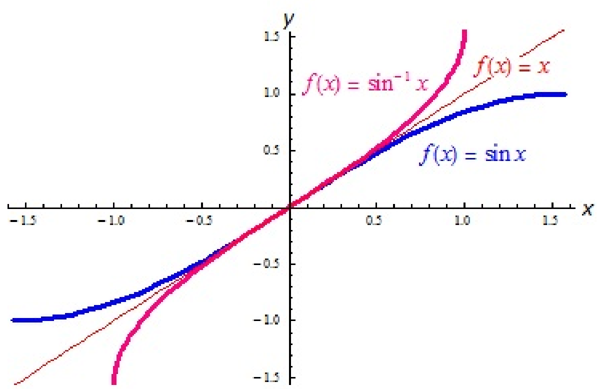
$$\text{For } -1 \le y \le 1, \arcsin(y) = x$$

means

$$\sin x = y$$

with

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$$



A similar argument allows us to define the inverse cosine, $\arccos x$ or $\cos^{-1}$ by,

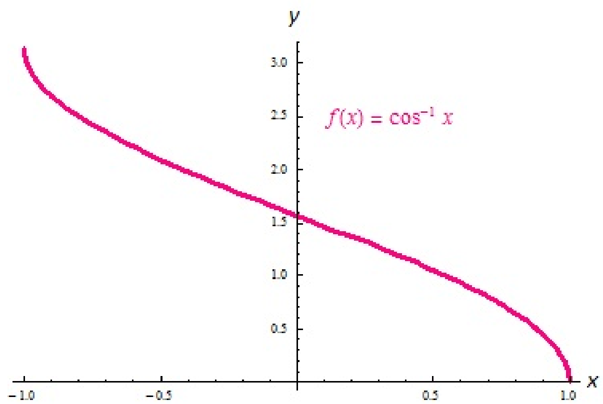
$$\text{For } -1 \le y \le 1, \arccos(y) = x$$

means

$$\cos x = y$$

with

$$0 \le x \le \pi$$



Finally, we have the inverse tangent. Setting $\tan x = y$ give infinitely many solutions, but just one with $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and so we define $\arctan$ or $\tan^{-1}$ by

$$\text{For any real number } y, \arctan(y) = x$$

means

$$\tan x = y$$

with

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$$