Lesson 1.4: *Logarithmic Functions*

These are simply the inverses of the exponential functions, $a^x, a > 1$.

Note that: if $a > 1$, a^x is increasing and so invertible (as it passes the horizontal line test).

Hence we can define the logarithm to the base of a of x, $\log_a x$ by:

$$\log_a x = c \text{ means } a^c = x$$

Note: $a^c = x$ means that $x > 0$ and so these functions naturally have domain $(0, \infty)$ (and range $\mathbb{R}$).

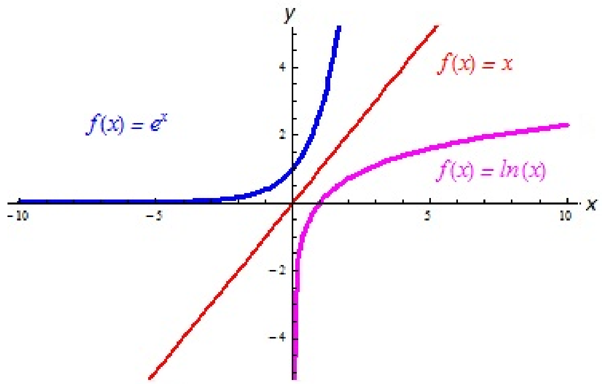
Note: $\log_a x$ makes no sense if x is zero or negative!!

The most important examples are $\log_{10} x = c$ which is given by:

$$\log_{10} x = c \text{ means } 10^c = x$$

and (much more importantly) $\log_e x$, called the natural logarithm of x and usually written $\ln(x)$ which is given by:

$$\ln x = c \text{ means } e^c = x$$



Example:

$$\log_{10} 1,000 = 3 \text{ as } 10^3 = 1,000 \text{ (as } c = 3)$$

$$\ln(e^2) = 2$$

$$\ln(e) = 1$$

$$\ln(1) = 0 \text{ as } e^0 = 1$$

Laws of Logarithms

Law 1: $\ln(ab) = \ln(a) + \ln(b)$

Law 2: $\ln(a/b) = \ln(a) - \ln(b)$

Law 3: $\ln(a^b) = b\ln(a)$

Law 4: $\ln(e^x) = x$

Law 5: $e^{\ln(x)} = x, x > 0$

Law 6: $\log_a(x) = \frac{\ln x}{\ln a}$

Law 7: $\ln(1) = 0$

Law 8: $\ln(e) = 1$

Examples:

Example 1:

Find t such that $2^t = 7$.

$$2^t = 7$$

Take the ln of both sides.

$$\ln(2^t) = \ln(7)$$

$$t\ln(2) = \ln(7) \qquad \text{(Law 3)}$$

$$t = \frac{\ln(7)}{\ln(2)}$$

$$\approx 2.81$$

Note: The book uses $\log_{10}$, but ln is more widely used.

Example 2:

Recall from Example 2 in *Lesson 2* where we had a sample of radioactive material with a half-life of 2 days. Find the time it take until all but 10% of the atoms have decayed. Now recall that we derived the formula

$$N(t) = N_0 2^{-t/2}$$

for the number of atoms as a function of t days.
Start with N_0 atoms. When only 10% are undecayed, we have $N_0/10$ radioactive atoms left.

Now set

$$N_0/10 = N_0 2^{-t/2}$$

and solve for t.

1: Divide by N_0

$$\frac{1}{10} = 2^{-t/2}$$

2: Take the ln of both sides.

$$\ln(\frac{1}{10}) = \ln(2^{-t/2})$$

$$= -(t/2)\ln(2) \qquad \text{(Law 3)}$$

$$-\ln(10) = -(t/2)\ln(2) \qquad \text{(Law 2, } \ln(\frac{1}{10}) = \ln(1) - \ln(b)\text{)}$$

$$\ln(10) = (t/2)\ln(2)$$

$$(t/2)\ln(2) = \ln(10)$$

$$(t/2) = \frac{\ln(10)}{\ln(2)}$$

$$t = \frac{2\ln(10)}{\ln(2)} \text{days}$$

Example 3:

The ozone in the ozone layer decays due to the steady release of CFCs according to the formula

$$Q(t) = Q_0 e^{-0.0025t}$$

where t is measured in years. Find the half-life of the ozone, i.e. the time it takes for half of the ozone to decay.

To find the half-life, t, we set $Q = Q_0/2$ to get:

$$
\begin{aligned}
\frac{Q_0}{2} &= Q_0 e^{-0.0025t} \\
\frac{1}{2} &= e^{-0.0025t} \\
\ln(\frac{1}{2}) &= \ln(e^{-0.0025t}) \\
&= -0.0025t && \text{(Law 4)} \\
t &= \frac{\ln \frac{1}{2}}{-0.0025} \\
&= \frac{-\ln(2)}{-0.0025} && \text{(Law 2)} \\
&= \frac{\ln(2)}{0.0025} \\
&= 400 \ln(2) \\
&\approx 277 years
\end{aligned}
$$

In the following example, we need to find the growth constant from knowing the quantity at two different times.

Example 4:

The population of Kenya was 19.5 million in 1986. Find a formula for the population as a function of time (assuming the population grows exponentially).

Let $P(t)$ be the population (in millions) where t is measured in years, starting at 1984.

Have $P = P_0 e^{kt} = 19.5e^{kt}$ (initial population in 1984). We need to find k, the growth constant.

Well, in 1986, $t = 2$, and we have 21.2 million, so:

$$
\begin{aligned}
21.2 &= 19.5e^{k2} \\
e^{2k} &= \frac{21.2}{19.5} \\
&\approx 1.087 \\
\ln(e^{2k}) &\approx \ln(1.087) \\
2k &\approx \ln(1.087) && \text{(Law 4)} \\
k &\approx 0.042
\end{aligned}
$$

Hence $P(t) = 19.5e^{0.042t}$.

Note:
The assumption of exponential growth in this problem is actually rather unrealistic.

Example 5:

Give a formula for the inverse of the following function (i.e. solve for t in terms of P).

$$P = f(t) = 67.38(1.026)^t$$

Take ln:

$$\begin{aligned}
\ln(P) &= \ln(67.38(1.026)^t) \\
&= \ln(67.38) + \ln(1.026)^t && \text{(Law 1)} \\
&= \ln(67.38) + t\ln(1.026) && \text{(Law 3)} \\
t\ln(1.026) &= \ln(P) - \ln(67.38) && \text{(Rearranging)} \\
t &= \frac{\ln(P)}{\ln(1.026)} - \frac{\ln(67.38)}{\ln(1.026)} \\
&\approx 38.96\ln(P) - 164.03
\end{aligned}$$

Note:
Again we used ln instead of $\log_{10}$.