LESSON 2.5: *The Second Derivative:*

Consider a function f with derivative f'. We can also differentiate f' to get the so-called second derivative f'' of f, ie $f'' = (f')'$ (the derivative of the derivative). Sometimes written $\frac{d^2f}{dx^2} = \frac{d}{dx}(\frac{df}{dx})$ or $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx})$ if $y = f(x)$.

WHAT DO DERIVATIVE TELL US?

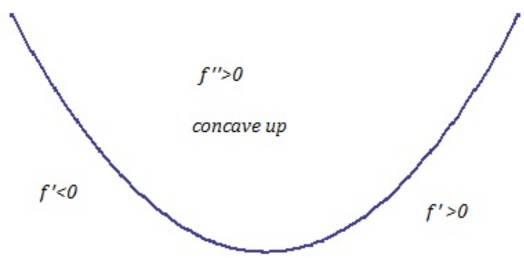
Recall that we have the following:

- If $f' > 0$ on an interval, then f is increasing on that interval.

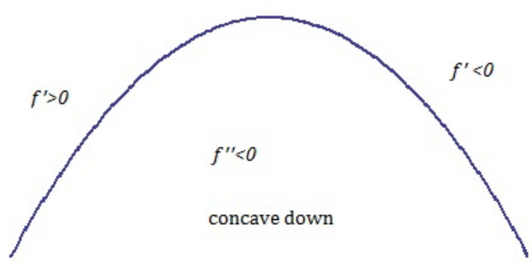
- If $f' < 0$ on an interval, then f is decreasing on that interval.

- If $f' = 0$ on an interval, then f is constant on that interval.

If we apply this result to f' instead of f, we get, for example, in the first case that if $f'' > 0$ on an interval then f' is increasing on that interval. Since f' represents the slope of the tangent line of the graph, it is plausible that this means that the graph of f must bend upwards - ie f is concave up.



Similar reasoning applies when $f'' < 0$.



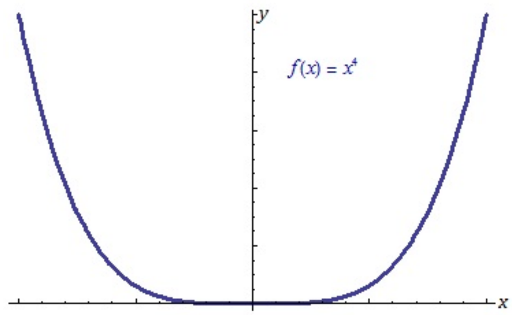
The full result is:

- If $f'' > 0$ on an interval, then f' is increasing on that interval, and f is concave up on that interval.

- If $f'' < 0$ on an interval, then f' is decreasing on that interval, and f is concave down on that interval.

- If $f'' = 0$ on an interval, then f' is constant on that interval, and f is linear on that interval.

<u>TRAP:</u> The converse of the first two parts is not (quite) true.

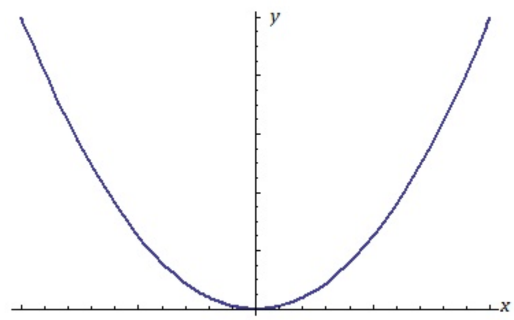


f is concave up on all of $\mathbb{R}$, but $f''(0) = 0$ - check this!!
What we *can* say is as follows:

- If the graph of f is concave up on an interval, then $f'' \geq 0$.

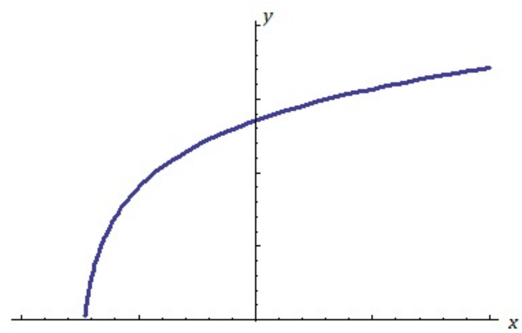
- If the graph of f is concave down on an interval, then $f'' \leq 0$.

Example 1: What do the following 3 graphs tell us about the second derivative?
a)



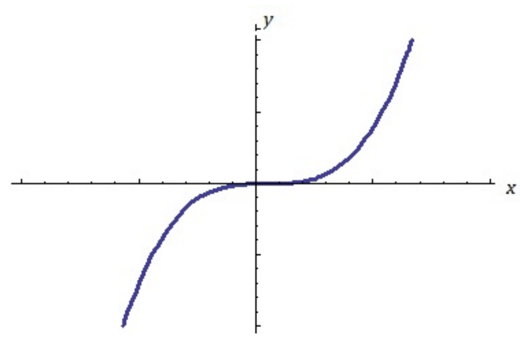
Concave up everywhere, so $f'' \geq 0$.

b)



Concave down everywhere so $f'' \leq 0$.

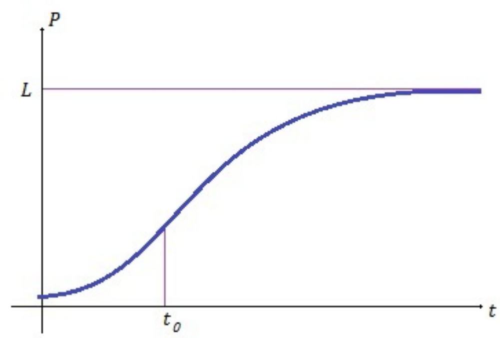
c)



Concave down for $x < 0$ so $f'' \leq 0$ for $x < 0$. Concave up for $x > 0$ so $f'' > 0$ for $x > 0$.
In the last example, the function changed from concave down to concave up as we passed through $x = 0$. A point where the concavity changes from up to down or vice versa is called a point of inflection.
Interpretation of the second derivative as a rate of change. Mention defence budged in 1985.

Example 2: Logistic Growth

Supposed a population of animals P grows with time t according the to the following graph:



We have a point of inflection at $t = t_0$ where the graph changes from concave up to concave down.

For $t < t_0$, $\frac{d^2P}{dt^2} \geq 0$ and the population growth rate $\frac{dP}{dt}$ appears to be growing.

For $t > t_0$, $\frac{d^2P}{dt^2} \leq 0$ and the population growth rate $\frac{dP}{dt}$ appears to be decreasing.

$\frac{dP}{dt}$ is actually at a maximum when $t = t_0$. As $t \to \infty, P \to L$ and the population appears to stabilize.

Example 3:
Tests on the C5 Chevy Corvette sports car gave the following table of data:

Time t (sec)	0	3	6	9	12
Velocity v (m/sec)	0	20	33	43	51

a) Estimate $\frac{dv}{dx}$ for the time intervals shown.

b) What can you say about $\frac{d^2v}{dt^2}$ over the period shown?

a) For each (short) time interval, $\frac{dv}{dt}$ is approximated by the average velocity: From $t = 0$ to $t = 3$

$$\frac{dv}{dt} \approx \text{Average rate of change of velocity} = \frac{v(3) - v(0)}{3 - 0} = \frac{20 - 0}{3 - 0} = 6.67m/s^2$$

The full table is:

Time interval t (sec)	0-3	3-6	6-9	9-12
Average acceleration v (m/s^2)	6.67	4.33	3.33	2.67

b) Since the values of $\frac{dv}{dt}$ are decreasing, we expect $\frac{d^2v}{dt^2} \leq 0$ and the graph of v against t should be concave down. That $\frac{dv}{dt} > 0$ means that the car is accelerating, but $\frac{d^2v}{dt^2} \leq 0$ means the acceleration is decreasing.

Velocity and Acceleration:

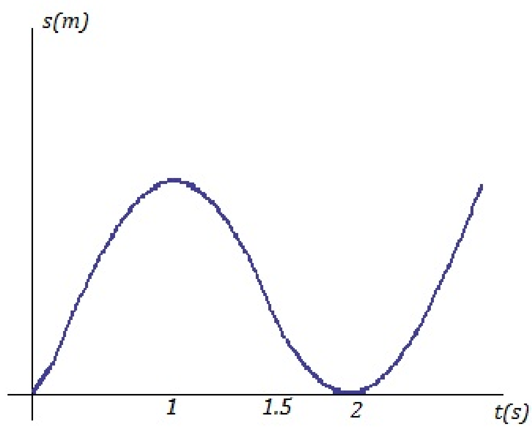
When a car is speeding up, we say it is accelerating. We define acceleration as the rate of change of velocity with respect to time. For an object with displacement $s(t)$ and velocity $v(t) = \frac{ds}{dt}$, the average acceleration over a time interval $[t, t+h]$ is

$$\text{Average Acceleration from } t \text{ to } t+h = \frac{v(t+h) - v(t)}{h}$$

while the instantaneous acceleration or acceleration at t is $a(t)$ where

$$a(t) = v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = s''(t)$$

Example 5: A particle is moving in a straight line, its acceleration is zero only once. Its displacement $s(t)$ has the following graph:



a) What is the particle moving to the right and when is it moving to the left?

b) When is the acceleration zero, negative, positive?

a) The particle is moving to the right whenever s is increasing. From the graph, this is for $0 < t < 1$ and $t > 2$. Similarly for $1 < t < 2$ s is decreasing, and the particle is moving the the left. Note that at $t = 2$, we are back where we started.

b) Particle is accelerating when the graph is concave up, decelerating when the graph is concave down. From the graph, this happens for $t > 1.5$ and $0 < t < 1.5$ respectively. Particle is not accelerating when concavity changes at the point of inflection located at $t = 1.5$.