LESSON 2.4: *Interpretations of the Derivative:*

Notation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$$

$$= \frac{dy}{dx}$$

$$= \frac{df}{dx}$$

$$= \frac{d}{dx}(y) \qquad \text{Leibniz notation}$$

Roughly speaking, can think of $\frac{dy}{dx}$ as a small change in y divided a (corresponding) small change in x.

In many real-world problems, the derivative has a particular meaning.

Using Units to interpret the Derivative:

$s = f(t)$ displacement

$v = \frac{ds}{dt} = f'(t)$ velocity

$a = \frac{dv}{dt} = v'(t) = \frac{d^2s}{dt^2} = f''(t)$ acceleration

Example:

$\frac{ds}{dt}\mid_{t=2} = 10m/s$, velocity of $10m/s$ at $t = 2s$

$\frac{dv}{dt}\mid_{t=2} = 1m/s$, acceleration of $1m/s$ at $t = 2s$

Example 1:

The cost, C in US dollars of building a house Aft^2 in area is given by the function $C(A)$. What is the practical interpretation of $f'(a)$?

$f'(A) = \frac{dC}{dA}$ and is measured in $\$/ft^2$. Can think of dC as the extra cost involved in adding a small amount dA to the area of the house. $\frac{dC}{dA}$ is then roughly the additional cost per square foot if we already have a house of area A, called marginal cost.

In practice, marginal cost $<$ average cost/square foot. Why?

Example 2:

The cost of extracting T tonnes of ore from a copper mine is $f(T)$ dollars. What does it mean to say $f'(2000) = 100$?

$f'(2000) = \frac{dC}{dT} = 100$. $\frac{dC}{dT}$ is measured in dollars/tonne. $f'(2000) = 100$ says, approximately, that when 2000 tonnes have already been extracted, the cost of the extracting the next tonne is about \$100.

Example 3:

If $q = f(p)$ gives the number of pounds of sugar produced when the price per pound is p dollars, then what are the units and the meaning of the statement $f'(3) = 50$?

Since $f'(3) = \lim_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$, $f'(3)$ has the same units as the difference quotient (average rate of change), which is pounds per dollar.

Therefore, $f'(3) = 50$ lbs/\$ means that when the price is 3 dollars the quantity produced is increasing by 50 lbs/\$. Thus, if the price increases by 1 dollar, the quantity increases by about 50 pounds.

Example 4:

Water is flowing through a pipe at $10ft^3/sec$.

Note: This doesn't really tell us how fast the water is moving through the pipe (since we don't know the cross-sectional area).

What it does tell us is that if $v(t)$ is the total volume which has flowed through the pipe by time t, then $\frac{dV}{dt} = 10ft^3/sec$.

Example 5:

Supposed $P = f(t)$ is the population of Mexico in millions, where t is the number of years since 1980. Interpret the following: a) $f'(6) = 2$, b)$f^{-1}(95.5) = 16$, c)$(f^{-1})'(95.5) = 0.46$.

a) $f'(6) = 2$ tells us that in 1986 (at $t = 6$), the population was increasing by 2 million per year.

b) $f^{-1}(95.5) = 16$ tells us that when the population was 95.5 million, t was 16, ie the year was 1996.

c) The units of $(f^{-1})'(p)$ are years/million. $(f^{-1})'(95.5) = 0.46$ tells us when the population was 95.5, it took about 0.46 years for the population to increase by 1 million.

Example 6:

The number of new subscriptions N to a newspaper in a month is a function of x, the amount in dollars spent on advertising in that month, so $N = f(x)$.

a) Interpret $f(250) = 180$, $f'(250) = 2$.
$f(250) = 180$ means if we spend \$250 a month on advertising, we get 180 new subscriptions.

$f'(250) = 2$ means that $\frac{dy}{dx} = 2$ when $x = 250$. Hence, if the amount spent is \$250, and goes up by \$1, then the number of new subscriptions goes up by about 2.

b) Using a), estimate $f(251)$ and $f(260)$. Which estimate is more reliable?

The tangent line $y = f(a) + f'(a)(x - a)$ gives a linear approximation to the graph of f near $x = a$. Here, $a = 250$, $f(a) = 180$, $f'(a) = 2$. So

$$f(251) \approx y = 180 + 2(251 - 250) = 182$$

and

$$f(260) \approx y = 180 + 2(260 - 250) = 200$$

The first approximation is clearly better since the linear approximation is better for values if x is near a.