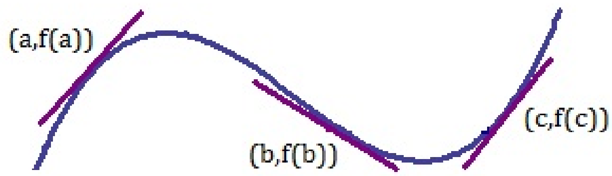
Consider a function, $f(x)$,



If we take tangent lines to the graph at different points, and then take the slopes of these lines, we get a new function called the derivative of f, f'.

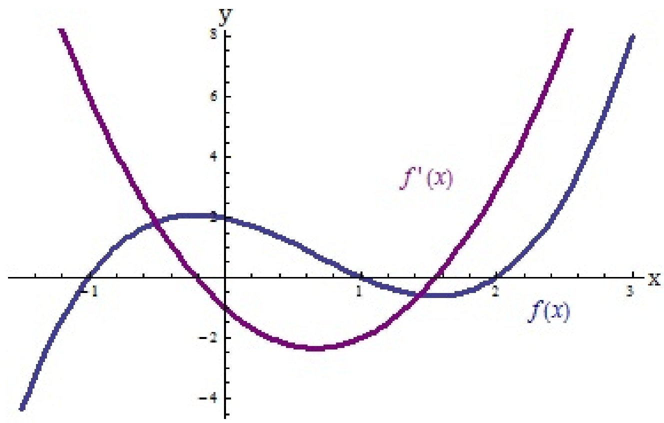
Definition:
For any function, f, we define the derivative function of f, f', by

$$\begin{aligned} f'(x) &= \text{Rate of change of } f \text{ at } x \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \end{aligned}$$

• limit of average rates of change over smaller and smaller intervals containing x.

For every x-value for which this limit exists, we say f is differentiable (diff) at x. If f is differentiable at every point in it's domain, we say f is differentiable everywhere.

Consider the following picture of a function and it's derivative.



What we notice is that when f is increasing, f' appears to be positive, and when f is decreasing, f' appears to be negative.

The general principle is:

- If $f' > 0$ on an interval, then f is increasing on that interval.

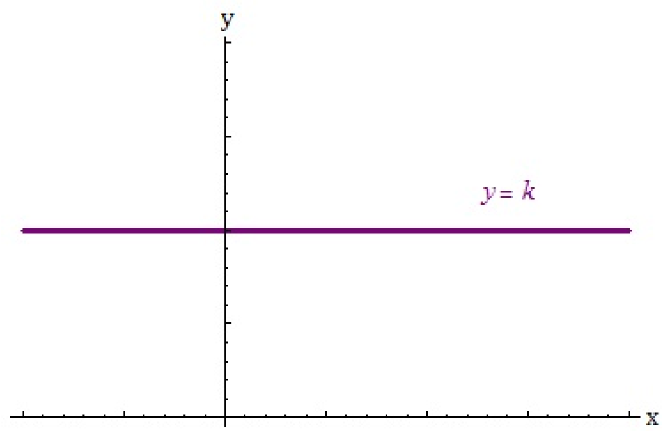
- If $f' < 0$ on an interval, then f is decreasing on that interval.

- if $f' = 0$ on an interval, then f is constant on that interval.

Derivative of Simple Functions:

Example 1: Constant Function

Graph of a constant function is a horizontal line, the derivative is zero everywhere.



For a constant function, $f(x) = k$, for any x,

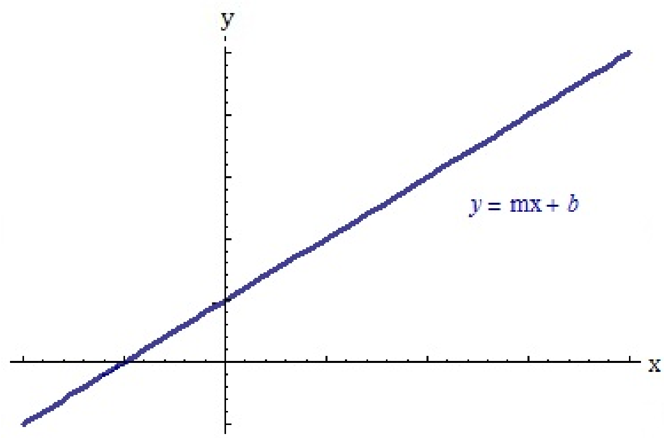
$$\text{Average rate of change} = \frac{f(x+h) - f(x)}{h} = \frac{k-k}{h} = 0$$

So

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$$

Example 2: Linear Function

The graph of a linear function $f(x) = mx + b$ is a straight line whose slope is m, the derivative is m everywhere.



For any k

$$\text{Average Rate of Change} = \frac{f(x+h) - f(x)}{h} = \frac{m(x+h) + b - (mx+b)}{h} = \frac{mh}{h} = m$$

So

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} m = m$$

Hence, $f'(x) = m$ everywhere.

Example 3: Power Functions

Find a formula for the derivative of $f(x) = x^2$. For any x

$$\begin{aligned}
\text{Average rate of change} &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^2 - x^2}{h} \\
&= \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \frac{2xh + h^2}{h} \\
&= 2x + h
\end{aligned}$$

So

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 2x + h = 2x$$

Hence $f'(x) = 2x$.

Example 4:

Find the derivative of $g(x) = x^3$

$$\frac{g(x+h) - g(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

Now $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, and so substituting,

$$\begin{aligned}
\frac{g(x+h) - g(x)}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{+3x^2h + 3xh^2 + h^3}{h} \\
&= 3x^2 + 3xh + h^2 \\
&\to 3x^2 + 0 + 0 \\
&= 3x^2 \text{ as } h \to 0
\end{aligned}$$

Hence $g'(x) = 3x^2$.

More generally, we can use the Binomial Theorem to show that for a positive integer n,

> If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, in particular, f is differentiable on all of $\mathbb{R}$.

or

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}, n \in \mathbb{N}, x \in \mathbb{R}}$$