§ 4.2, 4.4 Optimization Modelling

Global Maxima and Minima

- f has a global minimum at p if $f(p)$ is less than or equal to all other values of f.

- f has a global maximum at p if $f(p)$ is greater than or equal to all other values of f.

Many real-world problems can be reduced to maximizing or minimizing a function on an interval.

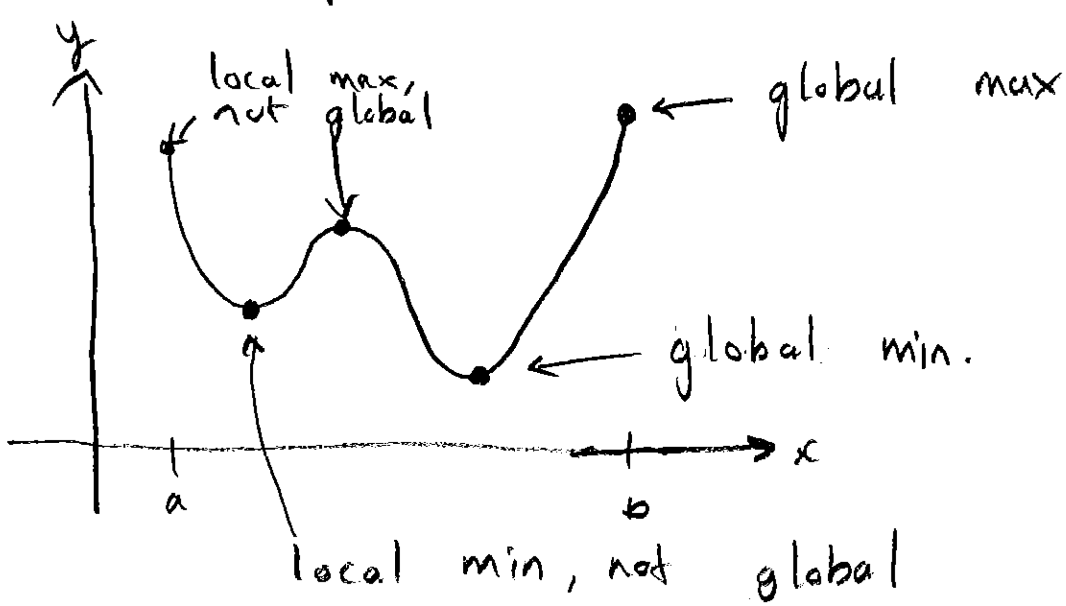
Closed Interval Problems

Working on a closed interval is nice because we always have a max/min.

Theorem 4.2 Extreme Value Theorem.

If f is continuous on a closed interval $[a, b]$, then f has a global maximum and a global minimum on $[a, b]$.

The max, min could be either at a critical point or at an endpoint.



<u>Moral</u> We need to find the values of f at any critical points and at the endpoints

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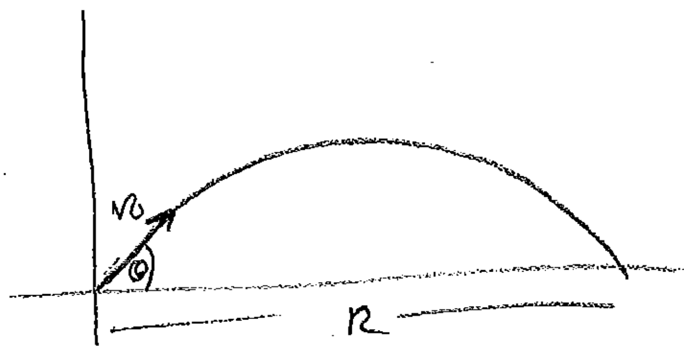
The largest of these values is the max and the smallest is the min.

Ex2 When an arrow is shot into the air, its horizontal range R is given by

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

where v_0 is the initial speed and θ_0 is the angle to the vertical at firing. Find the maximum value of R & for what angle it occurs.



Physically $0 \leq \theta \leq \frac{\pi}{2}$, so we maximize on this interval.

First find crit pts

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

So $$R'(\theta) = \frac{2 v_0^2 \cos(2\theta)}{g}$$

$R'(\theta) = 0 \Rightarrow \cos(2\theta) = 0$

$\Rightarrow \quad \theta = \ldots -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

Only soln in $[0, \frac{\pi}{2}]$ is $\underline{\frac{\pi}{4}}$.

$$R\left(\frac{\pi}{4}\right) = \frac{V_0^2 \sin\left(\frac{2\pi}{4}\right)}{g} = \frac{V_0^2 \cdot \sin\left(\frac{\pi}{2}\right)}{g} = \frac{V_0^2}{g},$$

End pts.

$R(0) = 0$

$R(\frac{\pi}{2}) = 0$

Hence the max. value of R is $\frac{V_0^2}{g}$ & this is attained for $\theta = \frac{\pi}{4} = 45°$.

General Procedure for Solving Optimization Problems - Seven Steps to Heaven

0. Read the question stupid!

1. Draw a (big) picture. Think about the problem in general terms first.

2. Determine the quantity to be optimized and find a formula for it - label your diagram accordingly.

3. Eliminate the unwanted variables. The information for this comes either from (rereading) the question or from the diagram. Commonly used things here are mensuration formulae, Pythagoras, trig (defn of sin, cos, tan).

4. Now that you have the quantity to be optimized, determine the interval on which it is to be optimized.

5. Find the global max/min as appropriate.

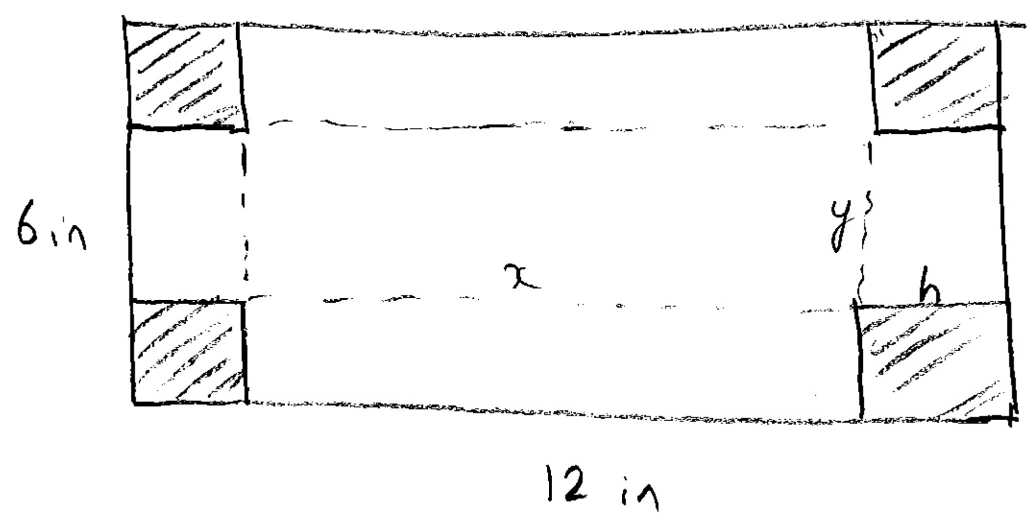
6. ~~Substitute to get your answer~~

7. Check with your intuition to see if your answer is reasonable.

Ex. Find the maximum volume of the box with an open top which can be made by cutting and folding a sheet of cardboard 6 in X 12 in as shown.

0!

1.!



2. Let the dimensions of the box be x, y, h as shown.

We want to maximize

$$V = xyh.$$

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3. Eliminate x, y (both depend on h).

From the picture

$$x + 2h = 12 \implies x = 12 - 2h$$

$$y + 2h = 6 \implies y = 6 - 2h$$

Substituting

$$\begin{aligned} V = V(h) &= (12-2h)(6-2h)h \\ &= (72 - 36h + 4h^2)h \\ &= 72h - 36h^2 + 4h^3. \end{aligned}$$

4. From purely physical considerations

$0 \leqslant h \leqslant 3.$

Hence we want to maximize

$$f(h) = 72h - 36h^2 + 4h^3$$

on the closed interval $[0, 6]$.

5.

Critical Points

$$f'(h) = 72 - 72h + 12h^2$$

$$= 12(6 - 6h + h^2)$$

$$f'(h) = 0 \Rightarrow h^2 - 6h + 6 = 0$$

Use quadratic formula

$$h = \frac{6 \pm \sqrt{36 - 24}}{2} = 3 \pm \sqrt{3}$$

Of these two, only $h = 3-\sqrt{3}$ is in $[0, 3]$.

<u>Using Mathematica</u>

$$f(3-\sqrt{3}) \approx 41.57$$

<u>Endpoints</u>

$f(0) = 0$ (flat box)

$f(3) = 0$ (thin vertical box)

Hence the max value of f on $[0,3]$ is 41.57 and it is attained at $h = 3-\sqrt{3} \approx 1.27$.

6. The max volume is then $41.57\ \text{in}^3$ (it is attained when the dimensions are
$$9.46\ \text{in} \times 3.46\ \text{in} \times 1.27\ \text{in}).$$

7. Seems reasonably reasonable!

Open Interval Problems

If we want to maximize/minimize a continuous function f on an open interval, then again we need to find the values of f at any critical points.

We also need to look at the behaviour of $f(x)$ as x approaches the endpoints of the interval (which could be $-\infty$ or ∞ if the interval is unbounded which often happens).

It can also be helpful to draw a graph of f to better see what is going on.

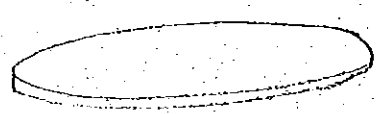
[illegible]imization and Modelling

Virgil's Aeneid.

Ex1. What are the dimensions of an aluminium can that holds 40 in^3 of juice & which uses the least amount of Aluminium. Assume the can is cylindrical and capped at both ends.

O!

1. Thinking about the problem in general terms

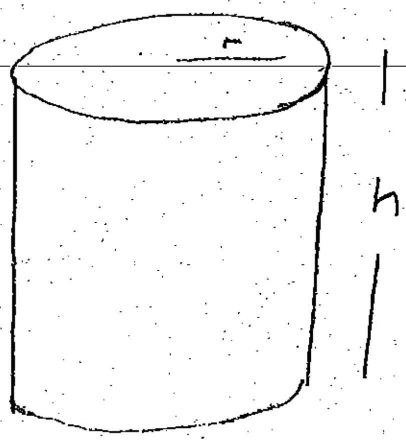


very flat can
- inefficient

very thin can - (probably) also inefficient.

Since the extremes seem to be inefficient, suggests that the best proportions for our can lie somewhere in the middle.

Let's make a picture.



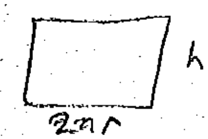
2.

Q. What is the quantity to be optimized?

A. The area of aluminium used.

Can has one curved side and two flat circular sides.

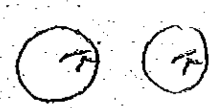
The area of the curved side is



$2\pi r h$ (imagine cutting along the seam and rolling flat)

The area of the two flat sides is

$\pi r^2 + \pi r^2 = 2\pi r^2$



Hence the total area to be mimized is

$A = 2\pi r h + \pi r^2$.

3.

We now come to what is (usually) the hardest part - eliminating the unwanted variable so that A is a fn of a single variable (r or h).

For this we need a relationship between r & h. Can we find one?

Well the volume must be 40 in^3.

Since the can is a cylinder, we have

$$V = \pi r^2 h = 40.$$

Hence $$h = \frac{40}{\pi r^2}.$$

Now subst this in our formula for A.

$$A = 2\pi r \cdot \frac{40}{\pi r^2} + 2\pi r^2$$
$$= \frac{80}{r} + 2\pi r^2.$$

4.

Q. What is the correct range for r?

A. $0 < r < \infty$.

So we want to minimise

$$A(r) = \frac{80}{r} + 2\pi r^2 \quad \text{on} \quad (0, \infty).$$

5.

Crit pts.

$$A'(r) = -\frac{80}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad 4\pi r = \frac{80}{r^2}$$

so

$$4\pi r^3 = 80$$

$$r^3 = \frac{20}{\pi}$$

$$r = \sqrt[3]{\frac{20}{\pi}} \simeq 1.85 \text{ in}.$$

Now

$$A''(r) = \frac{160}{r^3} + 4\pi$$

and for $r = \sqrt[3]{\frac{20}{\pi}}$, we have

$$A'(r) = \frac{160}{\left(\sqrt[3]{\frac{20}{\pi}}\right)^3} + 4\pi$$

$$= \frac{160}{\frac{20}{\pi}} + 4\pi$$

$$= 8\pi + 4\pi$$

$$= 12\pi > 0.$$

Hence by the 2nd deriv test.

$r = \sqrt[3]{\frac{20}{\pi}}$ is a local min.

Now we look at the endpts.

As $r \to 0$ $A(r) \to \infty$ and

as $r \to \infty$ $A(r) \to \infty$ also.

Hence the local min at $\sqrt[3]{\frac{20}{\pi}}$ is also a global min.

6. Here $h = \frac{40}{\pi r^2} = \frac{40}{\pi\left(\sqrt[3]{\frac{20}{\pi}}\right)^2} = \frac{40}{\pi\left(\frac{20}{\pi}\right)^{2/3}}$

$$= 2 \cdot \frac{20}{\pi \cdot \frac{20^{2/3}}{\pi^{2/3}}} = 2\left(\frac{20}{\pi}\right)^{\frac{1}{3}}$$

and

$$A = 2\pi r h + 2\pi r^2$$

$$= 2\pi \sqrt[3]{\frac{20}{\pi}} \cdot 2\sqrt[3]{\frac{20}{\pi}} + 2\pi\left(\sqrt[3]{\frac{20}{\pi}}\right)^2$$

$$= 6\pi\left(\frac{20}{\pi}\right)^{2/3} \approx 64.17 \text{ in}^2.$$

7.

Note that in the end

$$h = 2r$$

which makes our can as close to a sphere as possible.

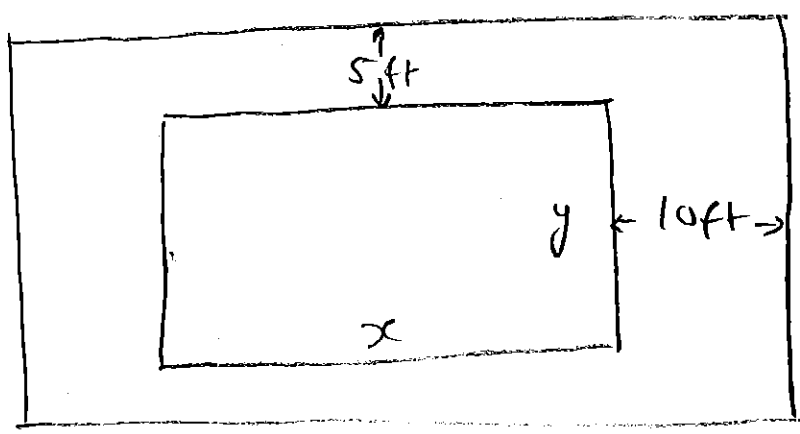
Since the sphere encloses the max. volume for the min surface area (e.g. soap bubbles), this answer seems reasonable.

Ex A rectangular swimming pool has an area of $1800 ft^2$. A deck is built around the pool which is 10ft wide at the ends and 5ft wide at the sides.

What proportions does the pool need to have in order to minimize the area of the deck?

0!

1.



2. Let x, y be the length, width of the pool.

The pool and deck form a rectangle of length $x + 2(10) = x + 20$ and width $y + 2(5) = y + 10$.

The area of the deck is then

$$A = (x+20)(y+10) - xy.$$

3. The area of the pool is 1800 ft^2, so

$$xy = 1800$$
$$y = 1800/x$$

Using this to eliminate y gives

$$A = A(x) = (x+20)\left(\frac{1800}{x} + 10\right) - 1800$$
$$= 1800 + \frac{36000}{x} + 10x + 200 - 1800$$
$$= \frac{36000}{x} + 10x + 200.$$

4. All we know about the length x is that it must be positive.

Hence we need to minimize

$$A(x) = \frac{36000}{x} + 10x + 200$$

on the open interval $(0, \infty)$.

5. Critical Points

$$A'(x) = -\frac{36000}{x^2} + 10$$

$$A'(x) = 0 \Rightarrow -\frac{36000}{x^2} + 10 = 0$$

$$\Rightarrow 10 = \frac{36000}{x^2}$$

$$1 = \frac{3600}{x^2}$$

$$x^2 = 3600$$

$$x = \pm 60$$

Take $x = +60$ for a critical pt. in $(0, \infty)$.

$$A''(x) = -\frac{72000}{x^3}$$

$A''(60) < 0$ so 60 is a local min by the second derivative test.

End behaviour

As $x \to 0_+$, $A(x) = \frac{36000}{x} + 10x + 200 \to \infty$

(because of $\frac{36000}{x}$)

As $x \to \infty$, $A(x) = \frac{36000}{x} + 10x + 200 \to \infty$

(because of $10x$).

Thus $x = 60$ is the global min.

6. The deck has minimum area when the pool has dimensions

$$x = 60\,\text{ft}, \quad y = \frac{1800}{60} = 30\,\text{ft}.$$

7. This answer is reasonable.