§ 4.8 Parametric Equations

Consider a particle moving in the plane. Its position as a function of time t is given by

$$x = f(t), \quad y = g(t)$$

or

$$(x, y) = (f(t), g(t))$$

where f and g are functions of time t.

This is an example of a <u>parametric curve</u> where t is called the <u>parameter</u>.

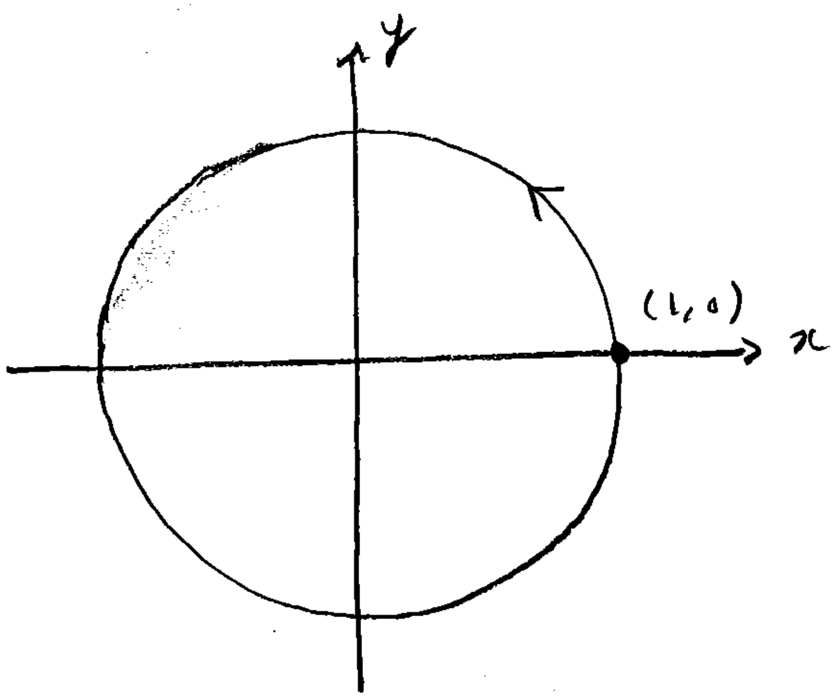
Ex $x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi.$

Here $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

and so we are on the unit circle.

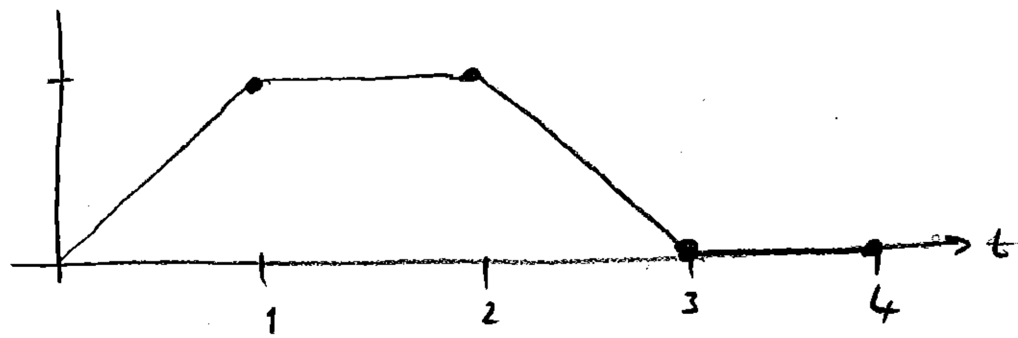
As t runs from 0 to 2π, we go once around the circle anticlockwise starting and finishing at

$$(\cos 0, \sin 0) = (\cos 2\pi, \sin 2\pi) = (1, 0).$$

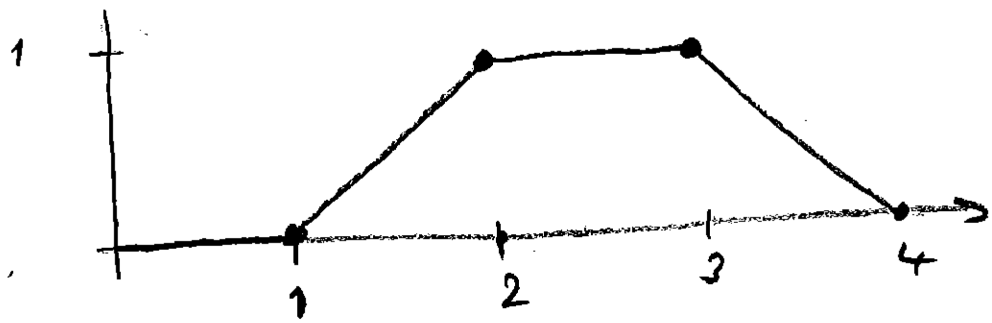


Ex 2

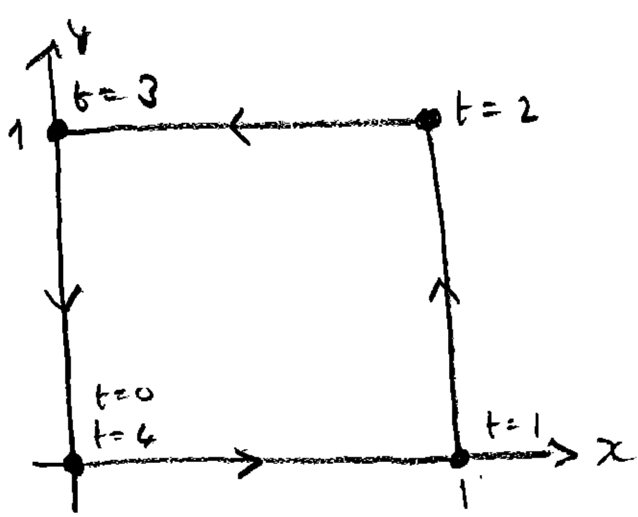
$f(t)$:



$g(t)$:



$(f(t), g(t))$, $0 \le t \le 4$ traces out a square as shown



<u>Ex 3</u> $x = \cos(3t)$, $y = \sin(3t)$

Since $\cos^2(3t) + \sin^2(3t) = 1$, this also traces out the unit circle.

However, it moves 3 times faster than the parametrization $(\cos t, \sin t)$

(e.g. it completes a whole revolution in $\frac{2\pi}{3}$ instead of 2π).

This example shows that there are many ways to parametrize the same curve.

Ex A straight line.

Here x and y change at a constant rate in t, i.e.

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b$$ for some constants a, b.

If we initially (at $t=0$) start at the point (x_0, y_0), then the parametric equation of the straight line is

$$x = x_0 + at, \quad y = y_0 + bt.$$

n.b. the slope of this line is b/a ($a \neq 0$).

Speed and Velocity

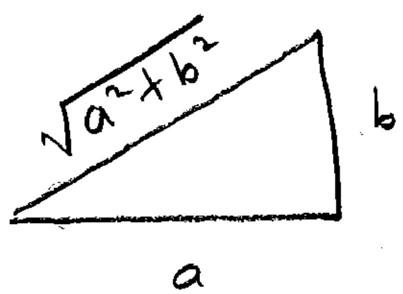
For an object moving as above in a straight line with constant velocity

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b$$

in one unit of time it travels

a units horizontally

---- b units vertically.



By Pythagoras, the distance travelled is $\sqrt{a^2+b^2}$ and the speed is

$$\text{Speed} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{\sqrt{a^2+b^2}}{1} = \sqrt{a^2+b^2}$$

For a general motion along a curve with (potentially) varying speed we define the instantaneous speed v by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$v_x = \dfrac{dx}{dt}$ is the velocity in the x-dir.

$v_y = \dfrac{dy}{dt}$ is the velocity in the y-dir.

The velocity vector $\vec{v}$ is given by

$$\vec{v} = v_x \vec{i} + v_y \vec{j}$$

where $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$ standard basis vectors in $\mathbb{R}^2$.

Ex A particle moves according to

$$x(t) = 2t + e^t, \quad y(t) = 3t - 4.$$

Find $v(1)$, $\vec{v}(1)$

ie. the speed and velocity vector when $t = 1$.

$$\frac{dx}{dt} = 2 + e^t, \quad \frac{dy}{dt} = 3$$

When $t = 1$, $v_x = 2 + e$, $v_y = 3$

so

$$\vec{v}(1) = (2+e)\vec{i} + 3\vec{j}.$$

while

$$v(1) = \sqrt{(v_x(1))^2 + (v_y(1))^2}$$

$$= \sqrt{(2+e)^2 + 3^2}$$

$$= \sqrt{13 + 4e + e^2} \approx 5.591$$

Ex A particle moves in the x-y plane according to

$$x(t) = 2t^3 - 9t^2 + 12t$$

$$y(t) = 3t^4 - 16t^3 + 18t^2.$$

a) At what times is the particle

i) stopped

ii) moving parallel to the x- or y-axis?

b) Find the speed of the particle at time t.

a)

Differentiate

$$\frac{dx}{dt} = 6t^2 - 18t + 12$$

$$\frac{dy}{dt} = 12t^3 - 48t^2 + 36t$$

Need to know when $\frac{dx}{dt} = 0$ and/or $\frac{dy}{dt} = 0$.

$$\frac{dx}{dt} = 6(t^2 - 3t + 2) = 6(t-1)(t-2)$$

so $\frac{dx}{dt} = 0$ when $t = 1$ or $t = 2$

$$\frac{dy}{dt} = 12t(t^2 - 4t + 3) = 12t(t-1)(t-3)$$

so $\frac{dy}{dt} = 0$ when $t = 0$, $t = 1$, $t = 3$.

i) Particle is stopped when both $\frac{dx}{dt}$ <u>and</u> $\frac{dy}{dt}$ are 0.

This happens when $t = 1$.

ii) Particle is moving parallel to the x-axis when $\frac{dy}{dt} = 0$ but $\frac{dx}{dt} \neq 0$.

This happens when $t = 0$, $t = 3$.

Particle is moving parallel to the y-axis when $\frac{dx}{dt} = 0$ but $\frac{dy}{dt} \neq 0$.

This happens when $t = 2$.

b)

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(6t^2 - 18t + 12)^2 + (12t^3 - 48t^2 + 36t)^2}$$

$$= 6\sqrt{4t^6 - 32t^5 + 89t^4 - 102t^3 + 49t^2 - 12t + 4}$$

Ex. A child is sitting on a ferris wheel of diameter 10m, making one revolution every 2 minutes. Find the speed of the child

a) Using geometry.

b) Using a parametrization of the motion.

a) Radius of circle $= 5m$.

Circumference $= 2\pi(5) = 10\pi m$

This is traversed in 2 minutes, so the speed is

$$\frac{\text{distance}}{\text{time}} = \frac{10\pi}{2} = 5\pi \approx 15.7 \text{ m/min}.$$

b) The radius of the wheel is 5 m and if we place the origin at the centre of the wheel, we have

$$x = 5\cos(\omega t), \quad y = 5\sin(\omega t)$$

where ω is chosen to reflect the fact that the period of revolution is 2 min, ie.

$$\omega(2) = 2\pi$$

$$\Rightarrow \quad \omega = \pi$$

and we have

$$x = 5\cos(\pi t), \quad y = 5\sin(\pi t).$$

Then $\frac{dx}{dt} = -5\pi\sin(\pi t)$, $\frac{dy}{dt} = 5\pi\cos(\pi t)$

and

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-5\pi\sin(\pi t)^2) + (-5\pi\cos(\pi t)^2}$$

$$= 5\pi\sqrt{\sin^2(\pi t) + \cos^2(\pi t)} = 5\pi \approx 15.7 \text{ m/s}$$

again.

Parametric Repesentatrons of Curves

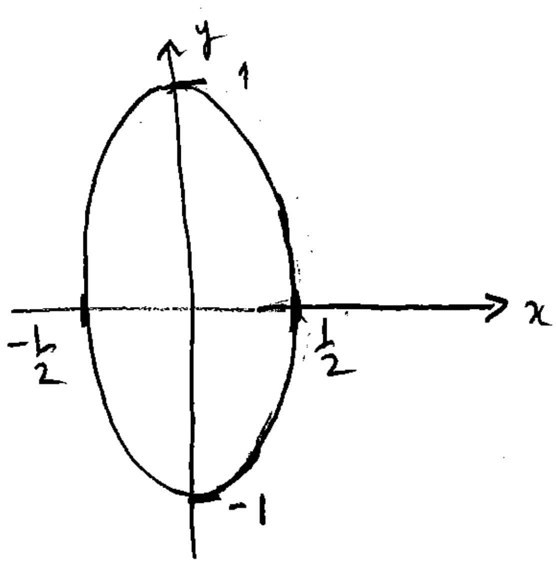
Ex. Find a parametrization of the ellipse $4x^2+y^2=1$.

Have $(2x)^2 + y^2 = 1$

so if we let $2x = \cos t$, then $y = \sin t$

$$x = \frac{\cos t}{2}, \quad y = \sin t$$

gives us a parametrization of the ellips.



Ex. Find a parametrization of the part of the hyperbola

$$x^2 - 4y^2 = 1$$

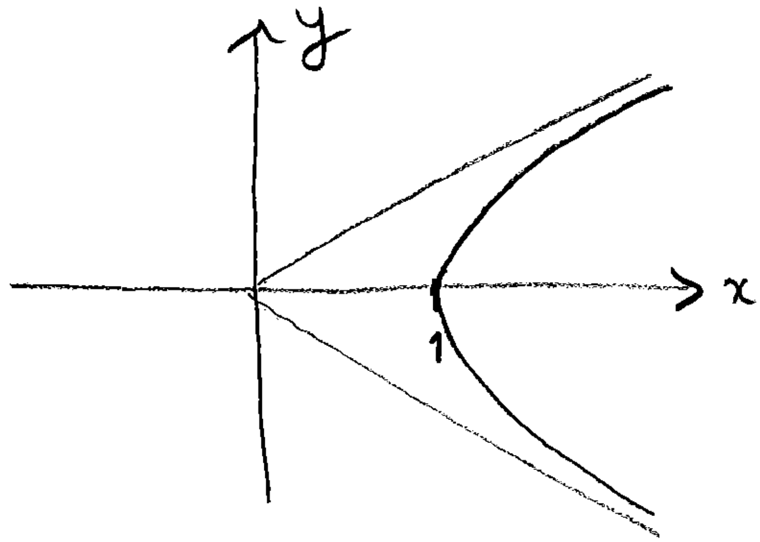
in the first and fourth quadrants.

If we let $x = \cosh t$, $2y = \sinh t$, then

$$x^2 - 4y^2 = 1.$$

Hence $x = \cosh t$, $y = \dfrac{\sinh t}{2}$ $\quad (t \in \mathbb{R})$

gives us a parametrization of this part of the hyperbola



Note that the graph of any function $y = f(x)$ can be parametrized by letting

$$x = t, \quad y = f(t).$$

e.g. $y = x^3 - x$ can be parametrized by letting

$$x = t, \quad y = t^3 - t.$$

Tangent Lines, Slope and Concavity of Parametric Curves

Ex Find the tangent line at the point (1, 2) to the curve defined by the parametric equation

$$x = t^3, \quad y = 2t$$

At time $t = 1$, $\quad x = 1^3 = 1$

$$y = 2(1) = 2$$

By differentiation,

$$v_x = 3t^2, \quad v_y = 2$$

and at $t = 1$

$$v_x = 3, \quad v_y = 2.$$

Tangent line passes through (1, 2) and so the eq^n is

$$x = 1 + 3t, \quad y = 2 + 2t.$$

We can also find the slope of parametric curves using the chain rule (works even when we can't eliminate t).

For $x = f(t)$, $y = g(t)$, if we know that y can be expressed as a function $h(x)$ of x, then by the chain rule (Leibniz form)

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

and if we divide by $\frac{dx}{dt}$ (provided $\frac{dx}{dt} \neq 0$), we get

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

One can remember this formula by cancelling the dt's.

We can find the second derivative $\frac{d^2y}{dx^2}$ by a similar method.

The same argument as above shows that if w is any differentiable fn of x, then

$$\frac{dw}{dx} = \frac{dw/dt}{dx/dt}.$$

If we then let $w = \frac{dy}{dx}$, then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dw}{dx}$$

$$= \frac{dw}{dt} \Big/ \frac{dx}{dt} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Ex For $x = \cos t$, $y = \sin t$ (unit circle)

find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ when $t = \frac{\pi}{4}$.

For $t = \frac{\pi}{4}$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and we are at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

So
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{\cos t}{\sin t} = -\cot t$$
$$= -1 \text{ when } t = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt} = \frac{\csc^2 t}{-\sin t}$$

$$= -\frac{1}{\sin^3 t} = -\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^3} = -2\sqrt{2}.$$