Chapter 3 - Shortcuts to Differentiation
Section 3.3 - The Product and Quotient Rules

Suppose we know the derivatives of two functions $f(x)$ and $g(x)$ and we want to calculate the derivative of the product $f(x)g(x)$.

Well, the average rate of change over $[x, x+h]$ is

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We can use a usual trick in this situation:
Add and subtract something halfway between the two terms (a bit of one and a bit of the other). In this case

$$f(x)g(x+h).$$

Using this, we get

$$\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

which we can split up into

$$\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}.$$

We now factor and number the different parts as follows:

$$\underbrace{\frac{f(x+h)-f(x)}{h}}_{\textbf{(1)}} \cdot \underbrace{g(x+h)}_{\textbf{(2)}} + \underbrace{f(x)}_{\textbf{(3)}} \cdot \underbrace{\frac{g(x+h)-g(x)}{h}}_{\textbf{(4)}}$$

where each colored number corresponds to the underlined piece of the expression directly above it. Let's look at what happens to each of these terms as we take a limit as $h \to 0$.

Note that (1) and (4) are just the average rates of change of f and g respectively over $[x, x+h]$, so as we let $h \to 0$, they tend to $f'(x)$ and $g'(x)$ respectively.

(3) is just a constant as far as h is concerned, so it just remains $f(x)$.

Finally, we come to (2). Recall that if a function is differentiable at a given point, then it is also continuous at that point. Hence g is continuous at x and since $x+h \to x$ as $h \to 0$, we see that (2), i.e. $g(x+h) \to g(x)$ as $h \to 0$.

This takes care of everything and in the limit we get

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = f'(x)g(x) + f(x)g'(x).$$

Thus we have shown the following theorem:

Theorem 3.3: The Product Rule. *If $u = f(x)$ and $v = g(x)$ are differentiable then so is $uv = f(x)g(x)$ and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

or (equivalently)

$$\frac{d}{dx}(uv) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

Example 1

a)

$$\begin{aligned}\frac{d}{dx}(x^2 e^x) &= \frac{d}{dx}(x^2) \cdot e^x + x^2 \cdot \frac{d}{dx}(e^x) \\ &= 2x \cdot e^x + x^2 \cdot e^x \\ &(x^2 + 2x)e^x \text{ (always remember to tidy up!)}\end{aligned}$$

b)

$$\begin{aligned}\frac{d}{dx}\big((3x^2 + 5x)e^x\big) &= \left(\frac{d}{dx}(3x^2 + 5x)\right) \cdot e^x + (3x^2 + 5x) \cdot \frac{d}{dx}(e^x) \\ &= (6x + 5)e^x + (3x^2 + 5x)e^x \\ &= (3x^2 + 11x + 5)e^x\end{aligned}$$

c)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^x}{x^2}\right) &= \frac{d}{dx}(x^{-2}e^x) \\
&= \left(\frac{d}{dx}(x^{-2})\right) \cdot e^x + x^{-2} \cdot \frac{d}{dx}(e^x) \\
&= -2x^{-3}e^x + x^{-2} \cdot e^x \\
&= (x^{-2} - 2x^{-3})e^x \\
&= \left(\frac{1}{x^2} - \frac{2}{x^3}\right) e^x \\
&= \left(1 - \frac{2}{x}\right) \frac{e^x}{x^2}
\end{aligned}$$

The Quotient Rule

A similar argument to that for products also gives us a formula for quotients of the form $\frac{f(x)}{g(x)}$ (where, of course, we avoid points where $g(x) = 0$).

Theorem 3.4: The Quotient Rule. *If $u = f(x)$ and $v = g(x)$ are differentiable then so is $\frac{u}{v} = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$ and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

or (equivalently)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - u\frac{dv}{dx}}{v^2}.$$

Example 2

a)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{5x^2}{x^3+1}\right) &= \frac{\frac{d}{dx}(5x^2)\cdot(x^3+1) - 5x^2\cdot\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\
&= \frac{10x(x^3+1) - 5x^2(3x^2)}{(x^3+1)^2} \\
&= \frac{10x(x^3+1) - 5x^2(3x^2)}{(x^3+1)^2} \\
&= \frac{10x^4 + 10x - 15x^4}{(x^3+1)^2} \\
&= \frac{10x - 5x^4}{(x^3+1)^2} \\
&= \frac{5x(2-x^3)}{(x^3+1)^2}
\end{aligned}$$

b)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{1+e^x}\right) &= \frac{\frac{d}{dx}(1)(1+e^x) - 1\cdot\frac{d}{dx}(1+e^x)}{(1+e^x)^2} \\
&= \frac{0-e^x}{(1+e^x)^2} \\
&= -\frac{e^x}{(1+e^x)^2}
\end{aligned}$$

c)

$$\begin{aligned}
\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{\frac{d}{dx}(e^x)\cdot x^2 - e^x\frac{d}{dx}(x^2)}{(x^2)^2} \\
&= \frac{e^x\cdot x^2 - e^x\cdot 2x}{x^4} \\
&= \frac{(x^2-2x)e^x}{x^4} \\
&= \left(1-\frac{2}{x}\right)\frac{e^x}{x^2}
\end{aligned}$$

Note: this is the same answer that we got using the product rule (as it should be).