Chapter 3 - Shortcuts to Differentiation
Section 3.2 - The Exponential Function

Differentiating Exponential Functions

Suppose we try to differentiate the exponential function

$$g(x) = a^x, \text{ for } a > 0.$$

We can proceed as follows:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \cdot \frac{a^h - 1}{h} \\ &= \left(\lim_{h\to 0} \frac{a^h - 1}{h}\right) \cdot a^x \\ &= k a^x \\ &= k g(x) \end{aligned}$$

where k is the constant

$$\lim_{h\to 0} \frac{a^h - 1}{h}$$

Note that k depends on a and one finds, for example, with $a = 2$, $k = .693$, while with $a = 3$, $k = 1.099$. Also, as a increases, k seems to increase.

It seems reasonable to guess (like in the Intermediate Value Theorem) that we can find a particular value e of a which lies between 2 and 3 so that

Result.

$$\frac{d}{dx}(e^x) = e^x.$$

In other words, e^x is its own derivative.

Back to Differentiating a^x

Instead of trying to find

$$\lim_{h \to 0} \frac{a^h - 1}{h}$$

directly, we use two tricks, namely, that

$$a = e^{\ln a} \text{ and } \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \text{ (a known fact).}$$

We had that

$$g'(x) = \left(\lim_{h \to 0} \frac{a^h - 1}{h} \right) \cdot g(x)$$

where $g(x) = a^x$. Now

$$
\begin{aligned}
\lim_{h\to 0} \frac{a^h - 1}{h} &= \lim_{h\to 0} \frac{e^{\ln a^h} - 1}{h} \\
&= \lim_{h\to 0} \frac{e^{h\ln a} - 1}{h} \\
&= \lim_{h\to 0} \ln a \left(\frac{e^{h\ln a} - 1}{h\ln a}\right) \text{ (multiplying top and bottom by } \ln a) \\
&= \ln a \lim_{h\to 0} \left(\frac{e^{h\ln a} - 1}{h\ln a}\right).
\end{aligned}
$$

Now let $t = h\ln a$. As $h \to 0$, $t \to 0$ and we can rewrite the above as

$$
\begin{aligned}
\ln a \cdot \lim_{t\to 0} \frac{e^t - 1}{t} &= \ln a \cdot 1 \text{ (using our known limit)} \\
&= \ln a.
\end{aligned}
$$

Hence

Result.

$$
\frac{d}{dx}(a^x) = (\ln a) \cdot a^x.
$$