Chapter 3 - Shortcuts to Differentiation
Section 3.1 - Powers and Polynomials

Constant Multiples

We saw that when we differentiated linear functions that

$$\tfrac{d}{dx}(mx+b)=m.$$

For example, if $f(x)=x+1$, then

$$\tfrac{d}{dx}(f(x))=1$$

$$\tfrac{d}{dx}(2f(x))=\tfrac{d}{dx}(2x+2)=2$$

$$\tfrac{d}{dx}(7f(x))=\tfrac{d}{dx}(7x+7)=7.$$

From this we see that the derivative of a constant multiple of f is the same constant times the derivative of f. This is true for any differentiable function.

Theorem 3.1: The Derivative of a Constant Multiple. *If f is differentiable and c is a constant, then*

$$\frac{d}{dx}[cf(x)]=cf'(x)$$

Proof:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0} c\frac{f(x+h)-f(x)}{h}\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{by Limit Law 1}\\ &= cf'(x).\square\end{aligned}$$

A similar result hold for sums and differences of differentiable functions.

Theorem 3.2: Derivative of Sum and Differences. *If f and g are differentiable functions, then*

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x),$$
$$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x).$$

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof: For differences

$$\begin{aligned}
\frac{d}{dx}[f(x)-g(x)] &= \lim_{h\to 0}\frac{[f(x+h)-g(x+h)-[f(x)-g(x)]}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\right] \\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \text{ by Limit Law 2} \\
&= f'(x)-g'(x). \square
\end{aligned}$$

Recall

Earlier, we saw that

$$\tfrac{d}{dx}(x^2) = 2x \text{ and } \tfrac{d}{dx}(x^3) = 3x^2,$$

and then we went on to say that for any power n,

$$\tfrac{d}{dx}(x^n) = nx^{n-1}$$

This works for any constant real number n (not just positive integers).

Example 1

$$\frac{d}{dx}\frac{1}{x^3} = \frac{d}{dx}(x^-3) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{2-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{d}{dx}\left(\frac{1}{x^{\frac{1}{3}}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{3}}\right) = -\frac{1}{3}x^{-\frac{1}{3}-1} = -\frac{1}{3}x^{-\frac{4}{3}} = -\frac{1}{3x^{\frac{4}{3}}}.$$

Example 2 Identify the power rule using the definition of a derivative for $n = -2$: show $\frac{d}{dx}(x^{-2}) = -2x^{-3}$.

Provided $x \neq 0$,

$$\begin{aligned}
\frac{d}{dx}(x^{-2}) &= \frac{d}{dx}\left(\frac{1}{x^2}\right) \\
&= \lim_{h\to 0}\left[\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x^2+2xh+h^2)}{x^2(x+h)^2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-2xh-h^2}{x^2(x+h)^2}\right] \\
&= \lim_{h\to 0}\frac{-2x-h}{x^2(x+h)^2} \text{ okay as } h \neq 0 \text{ for limit at } 0
\end{aligned}$$

$$= \frac{\lim_{h\to 0}(-2x-h)}{\lim_{h\to 0} x^2(x+h)^2} \text{ by Limit Law 4}$$

$$= \frac{-2\lim_{h\to 0} x - \lim_{h\to 0} h}{\lim_{h\to 0} x^2(\lim_{h\to 0}(x+h))^2} \text{ by Limit Laws 1, 2, and 3}$$

$$= \frac{-2-0}{x^2 \cdot x^2} \text{ by Limit Laws 5 and 6}$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

$$= -2x^{-3}.$$

Derivatives of Polynomials

Now that we can differentiate powers, constant multiples, and sums, we can differentiate any polynomial. If

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

is any polynomial, then

$$p'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ... + a_1.$$

Note that the derivative of a polynomial of degree n is another polynomial of degree $n-1$.

Example 3

a) $$\frac{d}{dx}(5x^2+3x+2) = 5\frac{d}{dx}x^2 + 3\frac{d}{dx}(x) + \frac{d}{dx}(2)$$
$$= 5 \cdot 2x + 3 \cdot 1 + 0$$
$$= 10x + 3$$

b) $$\frac{d}{dx}\left(\sqrt{3}x^7 - \frac{x^5}{5} + \pi\right) = \sqrt{3}\frac{d}{dx}(x^7) - \frac{1}{5}\frac{d}{dx}(x^5) + \frac{d}{dx}(\pi)$$
$$= \sqrt{3} \cdot 7x^6 - \frac{1}{5} \cdot 5x^4 + 0$$
$$= 7\sqrt{3}x^6 - x^4$$

Example 4

a) $$\frac{d}{dx}\left(5\sqrt{x}-\frac{10}{x^2}+\frac{1}{2\sqrt{x}}\right)=\frac{d}{dx}\left(5x^{\frac{1}{2}}-10x^{-2}+\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$=5\cdot\frac{d}{dx}(x^{\frac{1}{2}}-10\frac{d}{dx}(x^{-2})+\frac{1}{2}\frac{d}{dx}(x^{-\frac{1}{2}})$$
$$=5\cdot\frac{1}{2}x^{-\frac{1}{2}}-10\cdot-2x^{-3}+\frac{1}{2}\cdot-\frac{1}{2}x^{-\frac{3}{2}}$$
$$=\frac{5}{2\sqrt{x}}+\frac{20}{x^3}-\frac{1}{4x^{\frac{3}{2}}}$$

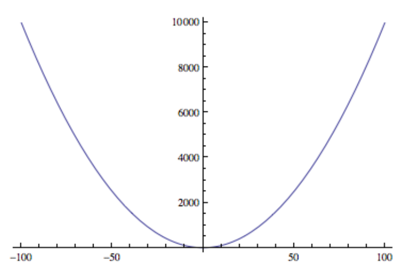
b) $$\frac{d}{dx}(0.1x^3+2x^{sqrt2})=0.1\frac{d}{dx}(x^3)+2\frac{d}{dx}(x^{\sqrt{2}})$$
$$=(0.1)\cdot 3x^2+2\cdot\sqrt{2}x^{\sqrt{2}-1}$$
$$=0.3x^2+2\sqrt{2}x^{\sqrt{2}-1}$$

Example 5 Find the second derivative and interpret its sign for

$$\text{a) } f(x)=x^2,\ \text{b) } g(x)=x^3,\ \text{c) } h(x)=x^{\frac{1}{2}}.$$

$$\textbf{a}) \; f(x)=x^2,\ f'(x)=2x,\ f''=\frac{d}{dx}(2x)=2$$

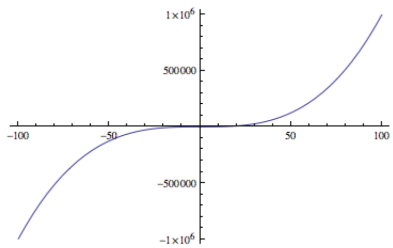
So $f''>0$ on $\mathbb{R}$ and so f is concave up on $\mathbb{R}$.



$$\textbf{b}) \; g(x)=x^3,\ g'(x)=3x^2,\ g''(x)=3\frac{d}{dx}(x^2)=3\cdot 2x=6x$$

$g''(x)<0$ for $x<0,$ so $g(x)$ is concave down for $x<0$

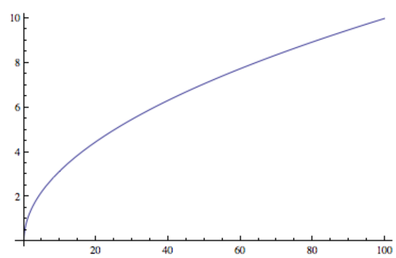
$g''(x)>0$ for $x>0$ so $g(x)$ is concave up for $x>0$



c) $h(x) = x^{\frac{1}{2}},\ h'(x) = \dfrac{1}{2}x^{-\frac{1}{2}},$

$$h''(x) = \frac{1}{2}\frac{d}{dx}(x^{-\frac{1}{2}}) = \frac{1}{2}\cdot -\frac{1}{2}\cdot x^{-\frac{3}{2}} = -\frac{1}{4}x^{-\frac{3}{2}}$$

So $h''(x) < 0$ for $x > 0$ and h is not defined for $x < 0$
so h is concave down on $(0, \infty)$.



Example 6 If the displacement of a body in meters is given as a function of time t in seconds by

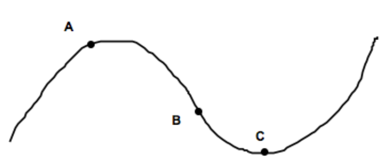
$$s = -4.9t^2 + 5t + 6$$

find the velocity and acceleration of the body at time t.

$$v = \frac{ds}{dt} = \frac{d}{dt}(-4.9t^2 + 5t + 6) = -9.8 + 5 \text{ m}\backslash s^2$$

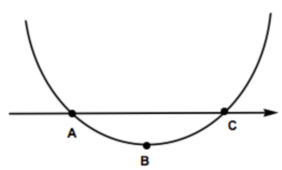
$$= a = \frac{dv}{dt} = \frac{d}{dt}(-9.8t + 5) = -9.8 \text{ m}\backslash s^2$$

Example 7 Consider the following cubic polynomial:



Graphically

Suppose we move along the curve from left to right. To the left of A the slope is positive, starts very positive and decreasing until the curve reaches A where slope is 0. Between A and C the slope is negative. Between A and B the slope is decreasing; it is at its most negative at B. Between B and C the slope is negative but increasing. At C the slope is 0. From C to the right the slope is positive and increasing. Hence the derivative looks as follows:



Algebraically

f is a cubic polynomial which goes to ∞ as $x \to \infty$, so

$$f(x) = ax^3 + bx^2 + cx + d$$

with $a > 0$. Hence

$$f'(x) = 3ax^2 + 2bx + c$$

whose graph is a parabola opening upwards.