§3.8 Hyperbolic Functions

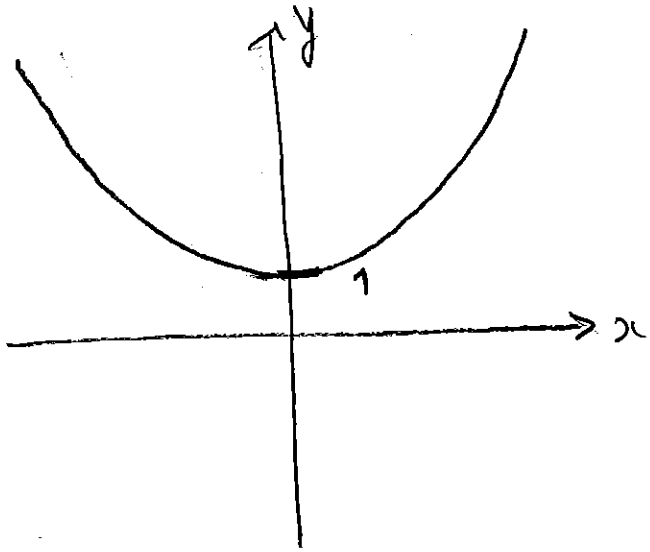
These are

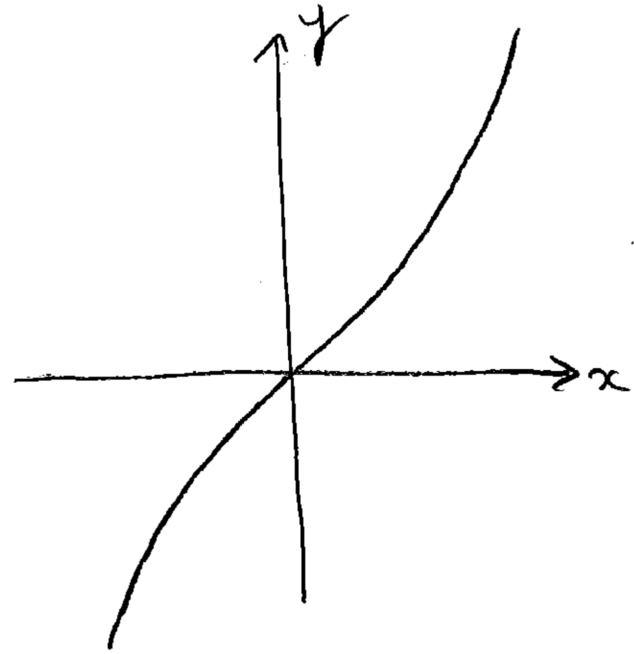
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Called the <u>hyperbolic cosine</u> and <u>hyperbolic sine</u> respectively.

Graphs





As suggested by the graphs

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

Also

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = -\sinh x$$

So $\cosh x$ is <u>even</u> while $\sinh x$ is <u>odd</u>.

As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$

so

$$\cosh x = \frac{e^x + e^{-x}}{2} \to \frac{\infty + 0}{2} = \infty$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \to \frac{\infty - 0}{2} = \infty$$

So

$$\lim_{x \to \infty} \cosh x = \infty$$

$$\lim_{x \to -\infty} \sinh x = -\infty$$

Similarly, as $x \to -\infty$, $e^x \to 0$ and $e^{-x} \to \infty$

So

$$\lim_{x \to -\infty} \cosh x = \infty$$

$$\lim_{x \to -\infty} \sinh x = -\infty$$

Identities involving $\cosh x$, $\sinh x$

Recall the regular trigonometric identity

$$\cos^2 x + \sin^2 x = 1.$$

We have an analagous identity for $\cosh x$, $\sinh x$.

First

$$\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$\sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4}$$

Both of these contain $\frac{e^{2x}}{4}$, $\frac{e^{-2x}}{4}$

and so if we subtract them, we get

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$- \left(\frac{e^{2x} - 2 + e^{-2x}}{4} \right)$$

$$= \frac{2}{4} - \left(\frac{-2}{4} \right) = 1$$

i.e.

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

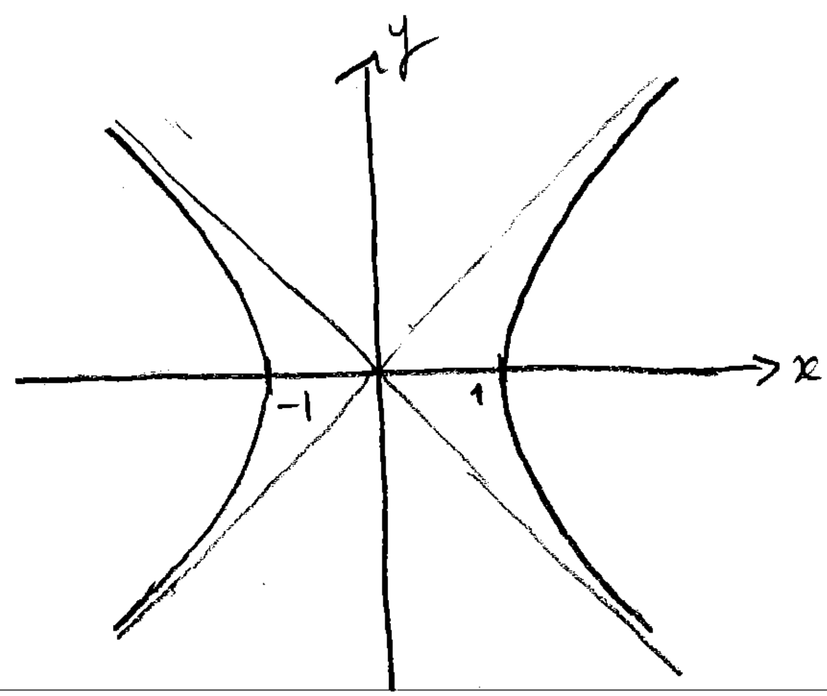
N.b. this identity is how the hyperbolic functions get the name 'hyperbolic'. Just as

$$\cos^2 x + \sin^2 x = 1$$

comes from the unit circle $x^2 + y^2 = 1$.

$\cosh^2 x - \sinh^2 x$ comes from the hyperbola

$$x^2 - y^2 = 1$$



We also have hyperbolic versions of other trigonometric functions.

e.g. Hyperbolic Tangent

$$\tanh x := \frac{\sinh x}{\cosh x}$$

Hyperbolic Cotangent

$$\coth x := \frac{\cosh x}{\sinh x}$$

Hyperbolic Secant

$$\operatorname{sech} x := \frac{1}{\cosh x}$$

Hyperbolic Cosecant

$$\operatorname{cosech} x := \frac{1}{\sinh x}$$

Differentiation

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{e^x - (-e^{-x})}{2}$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x.$$

So

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\sinh x) = \cosh x.$$

Compare with the derivatives of $\sin x$, $\cos x$.

Ex.

$$\frac{d}{dx}(\tanh x)$$

$$= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\frac{d}{dx}(\sinh x)\cosh x - \sinh x \frac{d}{dx}(\cosh x)}{\cosh^2 x}$$

by quotient rule

$$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}.$$ using $\cosh^2 x - \sinh^2 x = 1$

$(= \operatorname{sech}^2 x)$