MTH 141 Final Worksheet #3 Solutions

1. Evaluate the following definite integrals by interpreting them as (signed) areas.

i) $\int_0^2 \sqrt{4-x^2}\, dx$

Note that we are looking at the function $y = \sqrt{4-x^2}$ on the interval $[0,2]$. Also note that

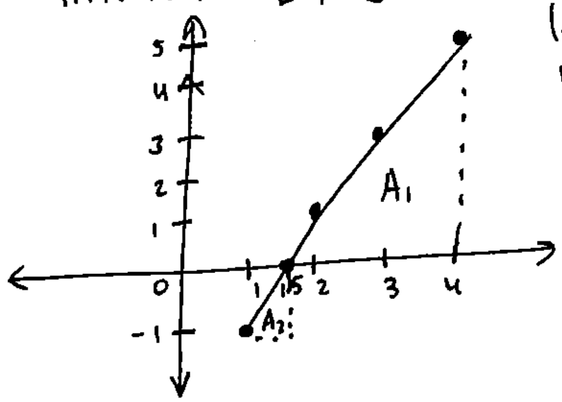
$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4 = 2^2,$$

so $\int_0^2 \sqrt{4-x^2}\,dx$ is the same as $\frac{1}{4}$ of the area of a circle centered at the origin with radius 2. Hence

$$\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi(2)^2 = \frac{1}{4}(\pi)4 = \pi.$$

ii) $\int_1^4 (-3+2x)\, dx$

Note that we are looking at the function $y = -3+2x$ on the interval $[1,4]$. We can graph this to see that we are looking for the area of a triangle of height 5 with base $\frac{5}{2}$ and the area of a triangle of height -1 and base $\frac{1}{2}$ [note that the height of -1 may not make sense intuitively, but we are looking for a signed area and our area is below the x-axis].



So $$\int_1^4 (-3+2x)\,dx = \frac{1}{2}(5)\left(\frac{5}{2}\right) + \frac{1}{2}(-1)\left(\frac{1}{2}\right)$$
$$= \frac{25}{4} - \frac{1}{4}$$
$$= \frac{24}{4} = 6.$$

2. If $\int_5^1 f(x)\,dx = 4$ and $\int_1^2 2f(x)\,dx = 3$, find $\int_2^5 f(x)\,dx$.

Note that $\int_1^5 f(x)\,dx = -\int_5^1 f(x)\,dx = -4$.

Note also that $\int_1^2 2f(x)\,dx = 3$

$$\Rightarrow \quad 2\int_1^2 f(x)\,dx = 3$$

$$\Rightarrow \quad \int_1^2 f(x)\,dx = \frac{3}{2}.$$

But $\int_1^5 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx$

$$\Rightarrow \quad -4 = \frac{3}{2} + \int_2^5 f(x)\,dx$$

$$\Rightarrow \quad -4 - \frac{3}{2} = \int_2^5 f(x)\,dx$$

$$\Rightarrow \quad -\frac{8}{2} - \frac{3}{2} = \int_2^5 f(x)\,dx$$

$$\Rightarrow \quad \frac{-11}{2} = \int_2^5 f(x)\,dx.$$

3. Calculate (without doing any calculations!)

i) $\int_{-10\pi}^{10\pi} \sin(x^7)\,dx = 0$

The trick here is to realize that since sin is an odd function, the areas above and below the x-axis will cancel each other out, on the interval $[-10\pi, 10\pi]$ leaving us with 0 for the answer.

ii) $\int_{100}^{100} e^{\sqrt{1+x^{27}}}\,dx = 0$

The trick here is to realize that there is no area on the interval $[100, 100]$ since this in in fact a single point, leaving us with 0 for the answer.

iii) $\int_0^3 (\cos^2 x + \sin^2 x)\,dx = 3$

The trick here is to realize that by a famous trigonometric identity, $\cos^2 x + \sin^2 x = 1$, and so

$$\int_0^3 (\cos^2 x + \sin^2 x)\,dx = \int_0^3 1\,dx = x\Big|_0^3 = 3-0 = 3.$$

4. Find the following indefinite integrals.

i) $\int \sin(3x)\,dx = \frac{1}{3}\cos(3x) + C$

ii) $\int \frac{(1+y)^2}{y}\,dy = \int \frac{(1+y)(1+y)}{y}\,dy$

$= \int \frac{1+2y+y^2}{y}\,dy$

$= \int \left(\frac{1}{y} + 2 + y\right) dy$

$= \ln|y| + 2y + \frac{y^2}{2} + C$

iii) $\int \left(t\sqrt{t} - \frac{1}{t\sqrt{t}}\right) dt = \int \left(t \cdot t^{\frac{1}{2}} - \frac{1}{t \cdot t^{\frac{1}{2}}}\right) dt$

$= \int \left(t^{\frac{3}{2}} - \frac{1}{t^{\frac{3}{2}}}\right) dt$

$= \int \left(t^{\frac{3}{2}} - t^{-\frac{3}{2}}\right) dt$

$= \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{t^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$

$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + C$

$= \frac{2}{5}t^{\frac{5}{2}} + 2t^{-\frac{1}{2}} + C$

$= \frac{2}{5}t^{\frac{5}{2}} + \frac{2}{t^{\frac{1}{2}}} + C$

$= \frac{2}{5}t^{\frac{5}{2}} + \frac{2}{\sqrt{t}} + C$

5. Use the Fundamental Theorem of Calculus (first part) to calculate.

i) $\int_0^2 (5x^4 - x^2 - 2x)\,dx = \left[\frac{5x^5}{5} - \frac{x^3}{3} - \frac{2x^2}{2}\right]\Big|_0^2$

$= \left[x^5 - \frac{x^3}{3} - x^2\right]\Big|_0^2$

$= \left(2^5 - \frac{2^3}{3} - 2^2\right) - \left(0^5 - \frac{0^3}{3} - 0^2\right)$

$= 32 - \frac{8}{3} - 4$

$= \frac{96}{3} - \frac{8}{3} - \frac{12}{3}$

$= \frac{76}{3}$

ii) $\int_0^{\frac{\pi}{3}} (\cos x + \sin(2x))\,dx = \left[\sin x - \frac{1}{2}\cos(2x)\right]\Big|_0^{\frac{\pi}{3}}$

$= \left[\sin\left(\frac{\pi}{3}\right) - \frac{1}{2}\cos\left(2\left(\frac{\pi}{3}\right)\right)\right] - \left[\sin(0) - \frac{1}{2}\cos(2\cdot 0)\right]$

$= \left[\frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{1}{2}\right)\right] - \left[0 - \frac{1}{2}(1)\right]$

$= \frac{\sqrt{3}}{2} + \frac{1}{4} + \frac{1}{2}$

$= \frac{\sqrt{3}}{2} + \frac{3}{4}$

iii) $\int_1^4 \left(e^{6x} + \frac{1}{x}\right) dx$

$= \left(\frac{1}{6}e^{6x} + \ln|x|\right)\Big|_1^4$

$= \left[\frac{1}{6}e^{6\cdot 4} + \ln 4\right] - \left[\frac{1}{6}e^{6\cdot 1} + \ln 1\right]$

$= \frac{1}{6}e^{24} + \ln 4 - \frac{1}{6}e^{6}$ since $\ln 1 = 0$.