MTH 141 Final Worksheet # 2

1. Differentiate each of the following.

i) $f(x) = \arctan(x^7)$

$$f'(x) = \frac{1}{1+(x^7)^2} \cdot 7x^6$$

$$= \frac{7x^6}{1+x^{14}}$$

ii) $g(x) = \ln(\arcsin(2x))$

$$g'(x) = \frac{1}{\arcsin(2x)} \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$= \frac{2}{\arcsin(2x)\sqrt{1-4x^2}}$$

iii) $h(x) = e^{x^2} \tan x$

$$h'(x) = e^{x^2} \sec^2 x + \tan x \, e^{x^2} \cdot 2x$$

$$= e^{x^2}\left(\sec^2 x + 2x \tan x\right)$$

iv) $k(x) = \dfrac{1+\sec x}{1-\sec x}$

$$k'(x) = \frac{(1-\sec x)(\tan x \sec x) - (1+\sec x)(-\tan x \sec x)}{(1-\sec x)^2}$$

$$= \frac{(\tan x \sec x)(1-\sec x + 1 + \sec x)}{(1-\sec x)^2}$$

$$= \frac{2\tan x \sec x}{(1-\sec x)^2}$$

v) $\ell(x) = 2^{\arcsin(x^3)}$

$$\ell'(x) = 2^{\arcsin(x^3)} \ln 2 \cdot \frac{1}{\sqrt{1-(x^3)^2}} \cdot 3x^2$$

$$= \frac{3 \ln(2)\ x^2 \cdot 2^{\arcsin(x^3)}}{\sqrt{1-x^6}}$$

$$= \frac{\ln(2^3)\ x^2\ 2^{\arcsin(x^3)}}{\sqrt{1-x^6}}$$

$$= \frac{\ln 8\ x^2\ 2 \arcsin(x^3)}{\sqrt{1-x^6}}$$

2. Use implicit differentiation to find $\frac{dy}{dx}$ for the relation

$$\sin(x+y) = (x-y)e^{y^2}$$

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = (x-y)\,e^{y^2}\cdot 2y\,\frac{dy}{dx} + e^{y^2}\left(1-\frac{dy}{dx}\right)$$

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = (x-y)e^{y^2}(2y)\frac{dy}{dx} + e^{y^2} - e^{y^2}\frac{dy}{dx}$$

$$\cos(x+y)\frac{dy}{dx} + e^{y^2}\frac{dy}{dx} - (x-y)e^{y^2}(2y)\frac{dy}{dx} = e^{y^2} - \cos(x+y)$$

$$\left[\cos(x+y) + e^{y^2} - (x-y)e^{y^2}(2y)\right]\frac{dy}{dx} = e^{y^2} - \cos(x+y)$$

$$\frac{dy}{dx} = \frac{e^{y^2} - \cos(x+y)}{\cos(x+y) + e^{y^2} - (x-y)e^{y^2}(2y)}$$

$$\frac{dy}{dx} = \frac{e^{y^2} - \cos(x+y)}{\cos(x+y) + e^{y^2} - (2yx - 2y^2)e^{y^2}}$$

$$\frac{dy}{dx} = \frac{e^{y^2} - \cos(x+y)}{\cos(x+y) + e^{y^2}(1 - 2yx + 2y^2)}$$

3. The number of eggs, E, a farmer sells each week depends on the price P, given in dollars, of the chicken feed.

a) Give the units of $E'(P)$.

$E'(P) = \frac{\Delta E}{\Delta P}$ so the units of $E'(P)$ are $\frac{\text{eggs}}{\text{dollars}}$

b) Explain the statement $E(100) = 312$ and $E'(100) = -2$.

- $E(100) = 312$ means if the farmer spends \$100 on chicken feed, then he will sell 312 eggs.
- $E'(100) = -2$ means when the farmer is spending \$100 on chicken feed, if he increases how much he is spending by a dollar, he will sell 2 less eggs.

4. a) Find the tangent line approximation for

$f(x) = \arctan x$ near $x = \frac{\pi}{4}$

$$f'(x) = \frac{1}{1+x^2} \qquad f\left(\frac{\pi}{4}\right) = \arctan\left(\frac{\pi}{4}\right)$$

$$f(x) - f\left(\frac{\pi}{4}\right) = f'(x)\left(x - \frac{\pi}{4}\right)$$

$$\arctan x - \arctan \frac{\pi}{4} = \frac{1}{1+x^2}\left(x - \frac{\pi}{4}\right)$$

$$\arctan x = \frac{1}{1+x^2}\left(x - \frac{\pi}{4}\right) + \arctan \frac{\pi}{4}$$

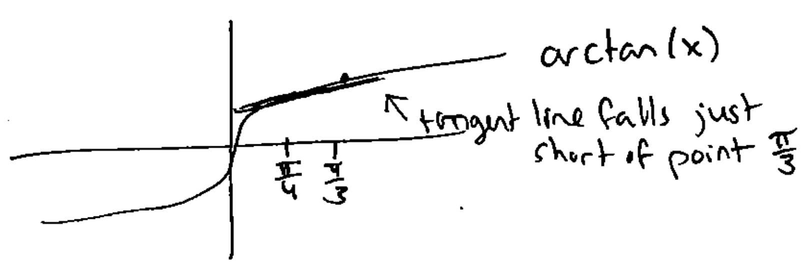
b) Use a) to approximate $\arctan \frac{\pi}{3}$.

$$\arctan \frac{\pi}{3} = \frac{1}{1+\left(\frac{\pi}{3}\right)^2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \arctan \frac{\pi}{4}$$

$$\approx .7906$$

c) Is the answer from b) larger or smaller than the actual value of $\arctan \frac{\pi}{3}$? Explain (without resorting to your calculators!)

The graph of $\arctan(x)$ is concave up on the interval $[0, \infty)$. Hence since $\frac{\pi}{4} < \frac{\pi}{3}$ we can expect our tangent line approximation to be smaller than the actual value of $\arctan\left(\frac{\pi}{3}\right)$.



5. Let $f(x) = x^4 - 4x^3 + 4x^2$

a) Find all the critical points of f and use the first derivative test to classify their type.

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$0 = 4x^3 - 12x^2 + 8x$$

$$0 = 4x(x^2 - 3x + 2)$$

$4x = 0$ or $x^2 - 3x + 2 = 0$

$x = 0$

$(x-2)(x-1) = 0$

$x - 2 = 0$ or $x - 1 = 0$

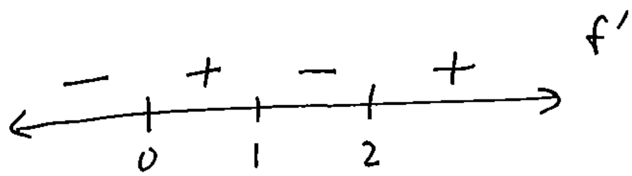
$x = 2$ $\qquad$ $x = 1$

The critical points are 0, 1, and 2.

$f(0) = 0$

$f(1) = 1$

$f(2) = 0$



So 0 is a local minimum, 1 is a local maximum, and 2 is a local minimum.

b) $f''(x) = 12x^2 - 24x + 8$

$$0 = 12x^2 - 24x + 8$$

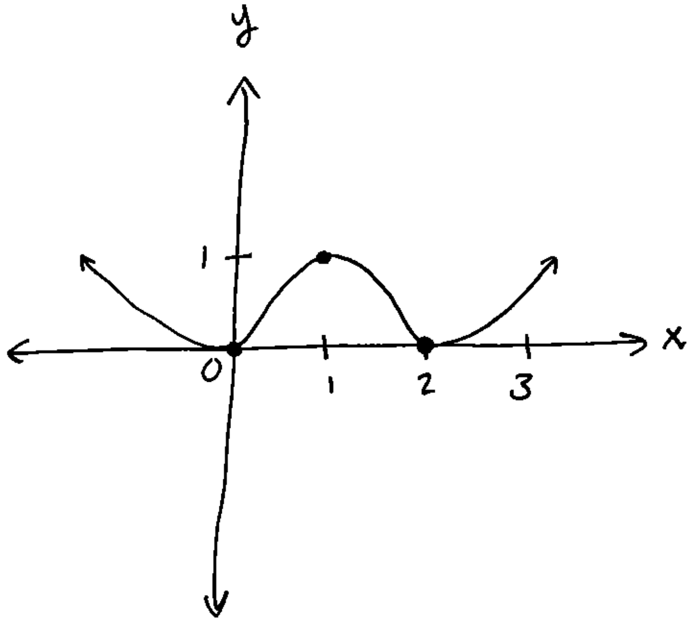
$$0 = 4(3x^2 - 6x + 2)$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} = \frac{6 \pm \sqrt{36 - 24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$= \frac{6 \pm 2\sqrt{3}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

So $1 + \frac{\sqrt{3}}{3}$, $1 - \frac{\sqrt{3}}{3}$ are the inflection points of $f(x)$.

c)



d) Find the max and min of f on the interval [.5,3].

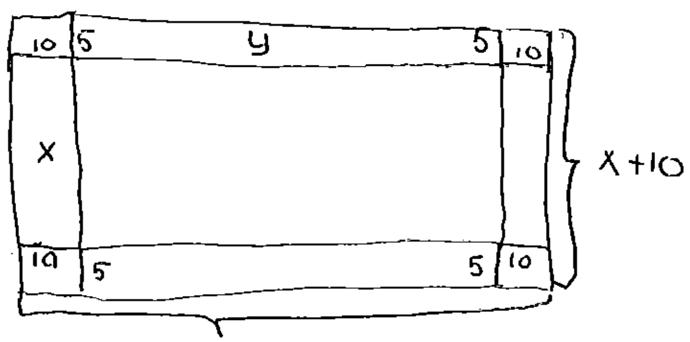
$f(.5) = .5625$

$f(3) = 9$

So, combining this information with what we know about our function evaluated at its critical points, we find that the maximum is 9 $\boxed{\text{occurring when } x=3}$ and the minimum is 0. $\boxed{\text{occurring when } x=2}$

#6. A rectangular swimming pool is to be built with an area of $1800\ ft^2$. The owner wants 5 ft wide decks ~~~~ ~~~~ along either side and 10 ft wide decks at the two ends. Find the dimensions of the smallest piece of property on which the pool can be built satisfying these conditions.

① Draw a picture:



$A = xy$

$P = 2x + 2y$

② We want to minimize the perameter of the pool and deck: $P = 2l + 2w$.

③ We know $xy = 1800$, and $P = 2(x+10) + 2(y+20)$.

We need P in terms of one variable. I choose x.

$y = \frac{1800}{x}$, substitute into P, get

$$P = 2(x+10) + 2\left(\frac{1800}{x} + 20\right).$$

④ $P = 2x + 20 + \frac{3600}{x} + 40$

$$= \frac{2x^2 + 60x + 3600}{x}$$

⑤ $P' = 0 = \frac{x(4x+60) - 2x^2 - 60x - 3600}{x^2}$

$$= \frac{4x^2 + 60x - 2x^2 - 60x - 3600}{x^2}$$

$$= \frac{2x^2 - 3600}{x^2}$$

$0 = 2x^2 - 3600$

$3600 = 2x^2$

$1800 = x^2$

$x \approx 42.43$ or $30\sqrt{2}$

* can't have negative length.

8. Calculate each of the following limits using l'Hôpital's rule.

i) $\lim_{x \to 0} \frac{\sin^2(2x)}{3x^2} \Rightarrow \frac{0}{0}$

L.H. $\lim_{x \to 0} \frac{2(\sin(2x))(\cos(2x))(2)}{6x}$

$= \lim_{x \to 0} \frac{4\sin(2x)\cos(2x)}{6x} \Rightarrow \frac{0}{0}$

L.H. $= \lim_{x \to 0} \frac{4[\cos(2x)(2)\cos(2x) + \sin(2x)(-\sin(2x))(2)]}{6}$

$= \frac{4}{6} \lim_{x \to 0} \left(2\cos^2(2x) - 2\sin^2(2x)\right)$

$= \frac{4}{6}\left(2\cos^2(2(0)) - 2\sin^2(2 \cdot 0)\right)$

$= \frac{4}{6}\left(2\cos^2(0) - 2\sin^2(0)\right)$

$= \frac{4}{6}\left(2(1) - 2(0)\right)$

$= \frac{4}{6}(2)$

$= \frac{4}{3}$.

ii) $\lim_{x \to -\infty} \frac{x^2}{e^{-x}} \Rightarrow \frac{\infty}{\infty}$

L.H. $\lim_{x \to -\infty} \frac{2x}{-e^{-x}} \Rightarrow \frac{-\infty}{-\infty}$

L.H. $\lim_{x \to -\infty} \frac{2}{e^{-x}}$

$= 0$

iii) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} \Rightarrow \frac{0}{0}$

L.H. $\lim_{x \to 1} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1}$

$= \lim_{x \to 1} \frac{1}{2\sqrt{x}}$

$= \frac{1}{2\sqrt{1}}$

$= \frac{1}{2}$

iv) $\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x}$ ← not in a form we can use L.H. on, so we must put it in such a form by finding a common denominator

$= \lim_{x \to 0} \frac{\sin x}{x \sin x} - \frac{x}{x \sin x}$

$= \lim_{x \to 0} \frac{\sin x - x}{x \sin x} \Rightarrow \frac{0}{0}$

L.H. $= \lim_{x \to 0} \frac{\cos x - 1}{x \cos x + \sin x} \Rightarrow \frac{0}{0}$

L.H. $= \lim_{x \to 0} \frac{-\sin x}{-x \sin x + \cos x + \cos x}$

$= \frac{-\sin(0)}{-0 \sin(0) + \cos(0) + \cos(0)}$

$= \frac{0}{0+1+1}$

$= \frac{0}{2}$

$= 0.$