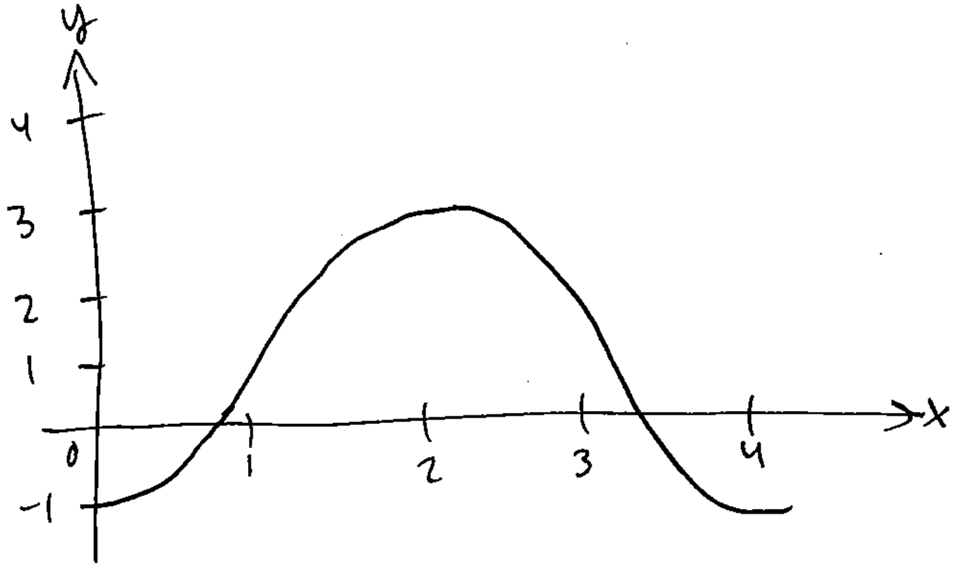
MTH 141 Final Worksheet #1 Solutions

1. Consider the trigonometric function $f(x)$ with the following graph.



a) Find the amplitude.

Recall: the amplitude A is the distance from the midpoint (1 in this case) to the highest or lowest point of the trig function (3 or -1, respectively, in this case). Hence $A = |1-3| = |-2| = 2$

b) Find the period.

Recall: The period P is the distance ~~it takes~~ for the trig function to start repeating itself. In this case, that distance is $P=4$

c) Give a formula for $f(x)$.

This function looks like an upside down cosine function so we will consider

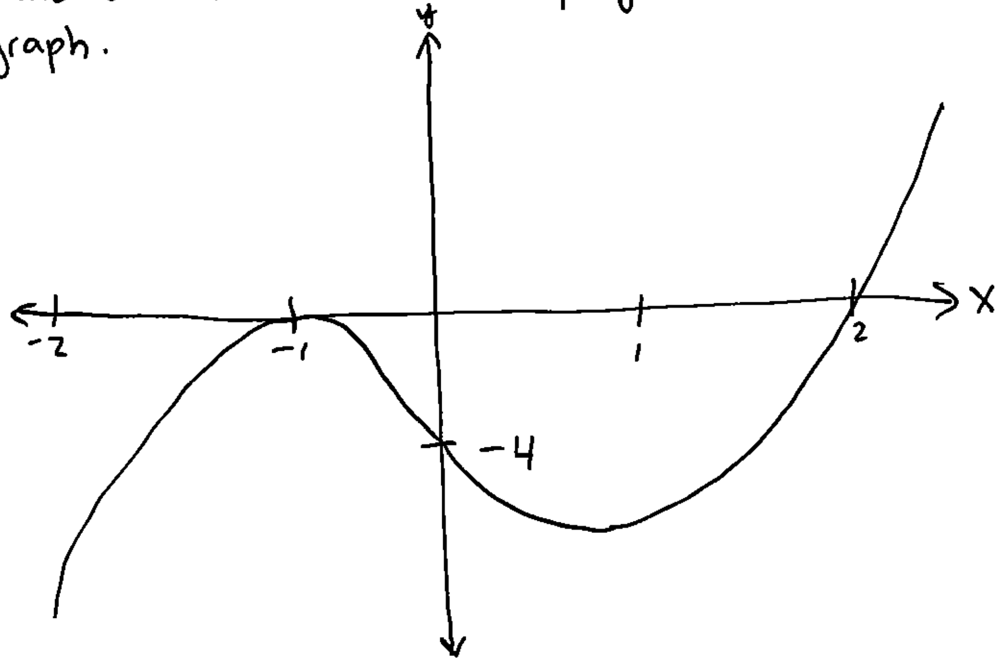
$$y = -A\cos(Bx + C) + D$$
$$y = -2\cos(Bx + C) + D$$
$$y = -2\cos\left(\tfrac{\pi}{2}x + C\right) + D$$
$$y = -2\cos\left(\tfrac{\pi}{2}x\right) + 1$$

since the graph has not been shifted left or right but is shifted up one unit

Recall:

$$P = \frac{2\pi}{B}$$
$$B = \frac{2\pi}{P} = \frac{2\pi}{4} = \frac{\pi}{2}$$

2. Find a formula for the polynomial with the following graph.



Note that the polynomial must have x-intercepts $(-1,0)$ and $(2,0)$ and y-intercept $(0,-4)$. Also, since as $x \to -\infty$, $y \to -\infty$ and as $x \to \infty$, $y \to \infty$, the leading coefficient is positive. So a function that satisfies these criterion is

$$f(x) = 2(x+1)(x-2)$$
$$= 2(x^2 - x - 2)$$
$$= 2x^2 - 2x - 4.$$

not necessary to multiply out, but we can see that we have the correct y-int in this form

Algebraically finding the leading coefficient k can be done by noting $f(x) = K(x+1)(x-2)$

$f(0) = K(0+1)(0-2) = -4$ since $(0,-4)$ y-intercept

$K(1)(-2) = -4$

$\frac{-2k}{-2} = \frac{-4}{-2}$

$K = 2$, which gives the polynomial above.

3. Let $g(x) = x^4 - 3x^2 - x + 1$. Show that there exists a c satisfying $1 < c < 2$ with $g(c) = 0$.

First note that $g(1) = 1^4 - 3(1)^2 - 1 + 1 = 1 - 3 - 1 + 1 = -2$
while $g(2) = 2^4 - 3(2)^2 - 2 + 1 = 16 - 3(4) - 2 + 1 = 16 - 12 - 2 + 1 = 3$.

Since $g(x)$ is a polynomial, $g(x)$ is continuous on $[1, 2]$. Hence the intermediate value theorem applies, and we know that $g(x)$ must cover every value between -2 and 3 on the interval $[1, 2]$ since $g(1) = -2$ and $g(2) = 3$. Therefore $\exists\, c \in (1, 2)$ such that $g(c) = 0$.

4. Evaluate each of the following limits or show that they do not exist. Do <u>not</u> use l'Hôpital's rule!

- $$\lim_{x \to 4} \frac{x-4}{x^2-16} = \lim_{x \to 4} \frac{x-4}{(x+4)(x-4)}$$

$$= \lim_{x \to 4} \frac{1}{x+4}$$

$$= \frac{1}{4+4}$$

$$= \frac{1}{8}$$

- $$\lim_{x \to 4} \frac{\sqrt{x}-2}{x-2} = \frac{\sqrt{4}-2}{4-2}$$

$$= \frac{2-2}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

- $$\lim_{x \to 2^-} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x \to 2^-} \frac{1}{x(x-2)} - \frac{1}{2(x-2)}$$

$$= \lim_{x \to 2^-} \frac{2-x}{2x(x-2)}$$

$$= \lim_{x \to 2^-} -\frac{(x-2)}{2x(x-2)}$$

$$= \lim_{x \to 2^-} -\frac{1}{2x}$$

$$= -\frac{1}{2(2)}$$

$$= -\frac{1}{4}$$

- $\lim_{x \to 0} \frac{\sin(2\theta)}{\theta}$ given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$= \lim_{x \to 0} \frac{2\sin(\theta)\cos(\theta)}{\theta}$

$= \left(\lim_{x \to 0} \frac{\sin(\theta)}{\theta}\right)\left(\lim_{x \to 0} 2\cos\theta\right)$

$= (1)(2\cos(0))$

$= 2(1)$

$= 2$

- $\lim_{x \to 1} \frac{x^2}{(x-1)}$ DNE.

Note that

$= \lim_{x \to 1^+} \frac{x^2}{x-1}$ while $\lim_{x \to 1^-} \frac{x^2}{x-1}$

$= \frac{\text{positive number}}{\text{very small positive number}}$ $= \frac{\text{positive number}}{\text{very small negative number}}$

$= (\text{positive number})(\text{very large positive number})$ $= (\text{positive number})(\text{very large negative number})$

$= -\infty$

$= \infty$

Since the left- and right-hand limits do not match, the general limit $\lim_{x \to 1} \frac{x^2}{x-1}$ DNE.

- $$\lim_{x \to 1} \frac{x-1}{|x-1|} \quad \text{DNE}$$

Note that

$$\lim_{x \to 1^+} \frac{x-1}{|x-1|}$$

$$= \frac{\text{small positive number}}{\text{small positive number}}$$

$$= (\text{small positive number})(\text{large positive number})$$

$$= \infty$$

while

$$\lim_{x \to 1^-} \frac{x-1}{|x-1|}$$

$$= \frac{\text{small negative number}}{\text{small positive number}}$$

$$= (\text{small negative number})(\text{large positive number})$$

$$= -\infty$$

Since the left-and right-hand limits do not match, the general limit $\lim_{x \to 1} \frac{x-1}{|x-1|}$ DNE.

5. Let $f(x) = e^x$. Calculate the average rate of change of f over each of the following intervals.

a) $[0,1]$ $\quad \dfrac{f(1)-f(0)}{1-0} = \dfrac{e^1 - e^0}{1} = e-1 \approx 1.718$

b) $[0,.1]$ $\quad \dfrac{f(.1)-f(0)}{.1-0} = \dfrac{e^{.1}-e^0}{.1} = \dfrac{e^{.1}-1}{.1} \approx 1.0517$

c) $[0,.01]$ $\quad \dfrac{f(.01)-f(0)}{.01-0} = \dfrac{e^{.01}-e^0}{.01} = \dfrac{e^{.01}-1}{.01} \approx 1.0050$

d) Guess the value of $f'(0)$ (which does exist!).

It appears that $f'(0) = 1$. (it does, and is easily verified).

6. Let $g(x)$ be given by the following table.

x	0	.1	.2	.3	.4
g(x)	.4	.3	.4	.5	.7

Use the table to estimate

a) $g'(.1) \approx \dfrac{\dfrac{g(.1)-g(0)}{.1-0} + \dfrac{g(.2)-g(.1)}{.2-.1}}{2} = \dfrac{\dfrac{.3-.4}{.1} + \dfrac{.4-.3}{.1}}{2}$

$= \dfrac{-1+1}{2} = \dfrac{0}{2} = 0$

b) $g'(.3) \approx \dfrac{\dfrac{g(.3)-g(.2)}{.3-.2} + \dfrac{g(.4)-g(.3)}{.4-.3}}{2} = \dfrac{\dfrac{.5-.4}{.1} + \dfrac{.7-.5}{.1}}{2}$

$= \dfrac{1+2}{2} = \dfrac{3}{2}$

7. Let

$$h(x) = \begin{cases} kx^2+2, & x \le 2 \\ 2x+k, & x > 2 \end{cases}$$

For what value of k is h continuous at $x=2$?

$$\lim_{x \to 2^+} 2x+k = 2(2)+k = 4+k$$

$$\lim_{x \to 2^-} kx^2+2 = k(2)^2+2 = 4k+2$$

In order for h to be continuous at $x=2$, we need $\lim_{x \to 2} h(x)$ to exist, so the left- and right-hand limits must be equal. So set

$$\begin{aligned} 4+k &= 4k+2 \\ -4k &\quad -4k \\ \hline 4-3k &= 2 \\ -4 &\quad -4 \\ \hline \frac{-3k}{-3} &= \frac{-2}{-3} \\ k &= \frac{2}{3}. \end{aligned}$$

So for $k = \frac{2}{3}$, h is continuous at 2.