§ 6.4 The Second Fundamental Theorem of Calculus

Suppose we consider an object like

$$F(x) = \int_0^x e^{-t^2}\,dt$$

Each different value of x gives us a different definite integral, so this is a function of x.

(Mysterious) Question — What happens if we try to differentiate such a function?

Theorem 6-2 Construction Theorem for Antds. (Second Fundamental Theorem of Calculus).

If f is cts. on an interval and a is any pt in that interval, then the F $\wedge$ F defined on the same interval by

$$F(x) = \int_a^x f(t)\,dt$$

is an antd. of f, i.e.

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

Idea of Proof

Try to differentiate $F(x)$

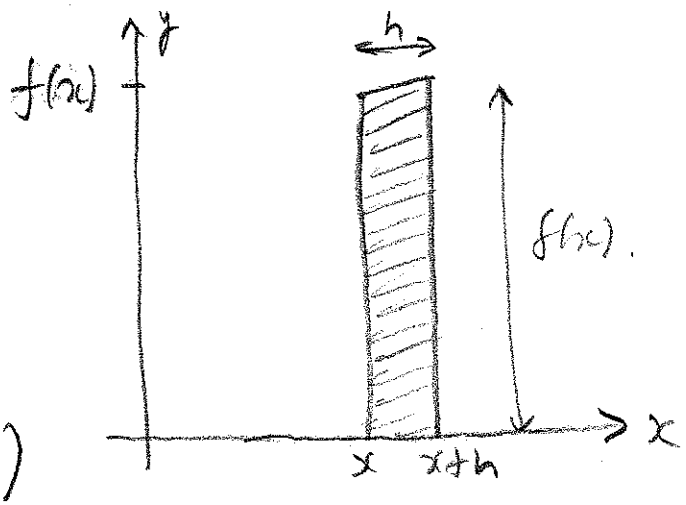
Newton quotient

$$\frac{F(x+h)-F(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$= \frac{\int_x^{x+h} f(t)\,dt}{h}$$

If h is small, then since f is cts., $f(t) \approx f(x)$ on $[x, x+h]$ and so $\int_x^{x+h} f(t)\,dt$ is approx the same as the (signed) area of a rectangle of width h and height $f(x)$.

So $$\frac{1}{h}\int_x^{x+h} f(t)\,dt$$



$$\approx \frac{1}{h}\cdot f(x)\cdot h = f(x)$$

(ok as $h \neq 0$ for a limit at 0)

Note that we can use this to prove FTOC (First part).

By Thm 6.2., if f is cts on $[a,b]$,

$$\int_a^b f(t)\,dt = F(b) = F(b) - F(a). \quad \left(\text{as } \int_a^a f(t)\,dt = 0\right)$$

Now if G is any other antid of f, then $G = F + C$ on $[a,b]$ for same const C and so

$$\begin{aligned} G(b) - G(a) &= (F(b) + C) - (F(a) + C) \\ &= F(b) - F(a). \end{aligned}$$

So $$\int_a^b f(t)\,dt = G(b) - G(a)$$

where G is <u>any</u> antid. of f.

This is FTOC.

Comparison between the two parts of FTOC.

Part 1. $\int_a^b f'(x)\,dx = f(b) - f(a)$.

– Integral of a Derivative.

Part 2. $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$

– Derivative of an Integral.

Ex.

Let $F(x) = \int_0^x e^{-t^2}\,dt.$

Then $F'(x) = e^{-x^2}.$

We can combine this with the rules of diff.

eg. $\frac{d}{dx}\left(x\,F(x)\right) = \frac{d}{dx}(x)\,F(x) + x\,\frac{d}{dx}\left(F(x)\right)$ product rule

$= F(x) + x\,F'(x)$

$= \int_0^x e^{-t^2}\,dt + x e^{-x^2}.$

Ex.

$$\frac{d}{dx}\left(\int_x^1 \sin t\,dt\right)$$

Here the x is on the 'wrong' limit but we can swap limits by taking a $-$ sign.

$$\frac{d}{dx}\left(\int_x^1 \sin t\,dt\right) = \frac{d}{dx}\left(-\int_1^x \sin t\,dt\right)$$

$$= -\frac{d}{dx}\left(\int_1^x \sin t\,dt\right)$$

$$= -\sin x \qquad \text{FTOC II.}$$

Ex. $\frac{d}{dx}\left(\int_0^{x^2} \sqrt{1+t^3}\, dt \right)$

Here the problem is that we have a fn of x rather than x in the limit. We deal with this by letting $u = x^2$ & using the chain rule.

Let $F(u) = \int_0^{x^2} \sqrt{1+t^3}\, dt$

By Chain rule (Leibniz Form).

$$\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}$$

$$= \frac{d}{du}\left(\int_0^{x^2} \sqrt{1+t^3}\, dt \right) \cdot \frac{d}{dx}(x^2).$$

$$= \frac{d}{du}\left(\int_0^u \sqrt{1+t^3}\,dt\right) \cdot 2x$$

$$= \sqrt{1+u^3} \cdot 2x \qquad \text{FTOC II}$$

$$= \sqrt{1+(x^2)^3} \cdot 2x$$

$$= 2x\sqrt{1+x^6}.$$