Constructing Antiderivatives

Recall that for a fn $f(x)$, another fn $F(x)$ is an antiderivative of f if $F'(x) = f(x)$.

$$F \xrightarrow{\text{differentiation}} f$$

$$f \xrightarrow{\text{anti differentiation}} F$$

Ex. x^2 is an antid of $2x$ since $\frac{d}{dx}(x^2) = 2x$.

However x^2+1, x^2-17, $x^2+\pi$ are also antids. of $2x$ as is x^2+c for any constant c.

Fact If F and G are both antids of the same fn f, then $G(x) = F(x) + C$ for some constant C.

Since all the antids. of a fn are so similar, this motivates the idea of trying to describe them all at once - the most general antiderivative of f.

If $F(x)$ is an antid. of $f(x)$, then the most general antid. of f is

$$F(x) + C$$

where C is an arbitrary, unspecified constant.

Another way of writing this is the indefinite integral

$$\int f(x)\,dx.$$

Note the difference between

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int f(x)\,dx$$

Definite Integral — Indefinite Integral.

Ex.

Since $\frac{d}{dx}(0) = 0$, $\int 0 \, dx = C$

$\frac{d}{dx}(kx) = k$, $\int k \, dx = kx + C$

$\frac{d}{dx}(x^2) = 2x$, $\int 2x \, dx = x^2 + C$.

More generally, if $n \neq -1$,

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{n+1 \; x^{(n+1)-1}}{n+1} = x^n, \quad \text{so}$$

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1}$$

What about $n = -1$?

If $x > 0$, then $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

If $x < 0$, $\ln x$ is undefined, but $\ln(-x)$ $= \ln|x|$ is and by the chain rule

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

Hence, if $x \neq 0$, $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ i.e. $\ln|x|$ is an antid. of $\frac{1}{x}$ and so

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

A few more...

$\frac{d}{dx}(e^x) = e^x$, so

$$\boxed{\int e^x \, dx = e^x + C}$$

$\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x$, so

$$\boxed{\int \sin x \, dx = -\cos x + C}$$

$\frac{d}{dx}(\sin x) = \cos x$, so

$$\boxed{\int \cos x \, dx = \sin x + C}$$

$$\frac{d}{dx}\left(\frac{1}{k}e^{kx}\right) = e^{kx}, \qquad k \neq 0$$

$$\Rightarrow \boxed{\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C, \quad k \neq 0.}$$

$$\frac{d}{dx}\left(-\frac{1}{k}\cos kx\right) = -\frac{1}{k}\cdot -\sin kx \cdot k = \sin kx, \quad k \neq 0.$$

$$\Rightarrow \boxed{\int \sin kx\,dx = -\frac{1}{k}\cos kx + C, \quad k \neq 0}$$

$$\frac{d}{dx}\left(\frac{1}{k}\sin kx\right) = \frac{1}{k}\cdot \cos kx \cdot k = \sin kx, \qquad k \neq 0$$

$$\Rightarrow \boxed{\int \cos kx\,dx = \frac{1}{k}\sin kx + C, \quad k \neq 0.}$$

Ex. Find $\int (3x + x^2)\,dx$

We know that $x^2/2$ is an antid of x

and $\frac{x^3}{3}$ is an antid. of x^2.

Now $\frac{d}{dx}\left(3 \cdot \frac{x^2}{2} + \frac{x^3}{3}\right)$

$= 3\frac{d}{dx}(x^2/2) + \frac{d}{dx}\left(\frac{x^3}{3}\right)$ by Laws of differentiation for sums & constant multiples

$= 3 \cdot x + x^2$

Hence

$$\int (3x + x^2)\,dx = \frac{3x^2}{2} + \frac{x^3}{3} + C.$$

This example shows that the laws for derivatives of sums and constant multiples also work in reverse when calculating antiderivatives.

More generally, we have

Theorem 6.1 Antiderivatives of Sums and Constant Multiples

1. $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

2. $\int c f(x)\,dx = c \int f(x)\,dx.$

In words.

1. An antid. of a sum (or difference) of two fns is the sum (or difference) of their antids.

2. An antid. of a constant multiple of a fn is the constant times the antid. of the fn.

Ex.

a) $$\int 8x^3 + \frac{1}{x}\,dx = 8\int x^3\,dx + \int \frac{1}{x}\,dx$$

$$= \frac{8x^4}{4} + \ln|x| + C$$

$$= \frac{x^4}{2} + \ln|x| + C.$$

b) $$\int 12e^{0.2t}\,dt = 12\left(\frac{1}{0.2}e^{0.2t}\right) + C$$

$$= 60e^{0.2t} + C$$

c) $$\int (\sin x + 3\cos(5x))\,dx$$

$$= \int \sin x\,dx + 3\int \cos(5x)\,dx$$

$$= -\cos x + \frac{3\sin(5x)}{5} + C$$

Note: When adding two or more indefinite integrals, one can always combine the constants they give rise to into one big constant.

Using Antiderivatives to Compute Definite Integrals

FTOC said, for a cts fn f

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F was such that $F' = f$, in other words F was an antid of f. So if we can find an antid. for f, we can use it to do definite integrals.

Ex.

$$\int_1^2 3x^2\,dx$$

Since $F(x) = x^3$ is an antid of $3x^2$, we have by FTOC that

$$\int_1^2 3x^2\,dx = F(2) - F(1) = 2^3 - 1^3 = 7$$

Instead of $F(2) - F(1)$, we often write

$$\left[x^3 \right]_1^2 = 2^3 - 1^3 = 7$$

which is shorter.