Theorem 5.3 Properties of Sums and Constant Multiples of the Integrand

Let f & g be cts. fns & let c be a constant

1. $$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

2. $$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$$

Integral of a sum or difference is the sum or difference of integrals.

Integral of a constant multiple is the constant multiple of the integral.

Pf for 1 with f+g

e.g. for 1., the left-hand sum for $\int f+g$ is

$$\sum_{i=0}^{n-1} (f(x_i) + g(x_i)) \Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x + \sum_{i=0}^{n-1} g(x_i) \Delta x.$$

Now let $n \to \infty$ and use the defn of the definite integral.

l.h.s. $\longrightarrow \int_a^b (f(x) + g(x))dx$

r.h.s. $\longrightarrow \int_a^b f(x)dx + \int_a^b g(x)dx.$

2. is similar.

Ex 2. $$\int_0^2 (1+3x)\,dx$$

$$= \int_0^2 1\,dx + \int_0^2 3x\,dx$$

$$= \int_0^2 1\,dx + 3\int_0^2 x\,dx.$$

1 has antid. x $\left(\frac{d}{dx}(x) = 1\right)$

x — — $\frac{x^2}{2}$ $\left(\frac{d}{dx}\left(\frac{x^2}{2}\right) = 1\right)$

So $$\int_0^2 dx + 3\int_0^2 x\,dx$$

$$= \left[x\right]_0^2 - 3\left[\frac{x^2}{2}\right]_0^2$$

$$= 2-0 - 3\left(\frac{2^2}{2} - \frac{0^2}{2}\right)$$

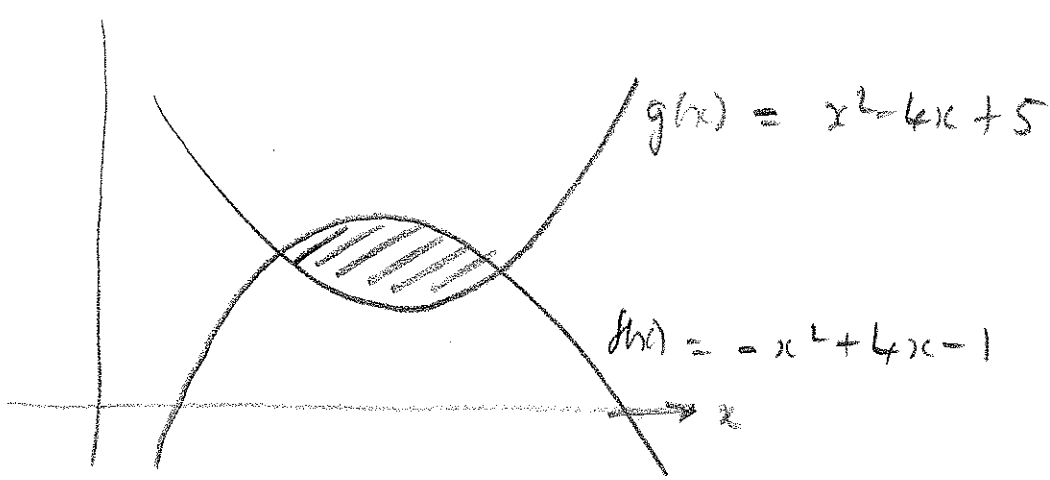
$$= 2 - 6 = -4.$$ (can also use area).

Area Between Curves

If $g(x) \leq f(x)$ for $a \leq x \leq b$, then the area between the graphs of f & g for $a \leq x \leq b$ is

$$A = \int_a^b (f(x) - g(x))\,dx.$$

Ex. Find the area of the region enclosed between the parabolas
$f(x) = -x^2 + 4x - 1$, $g(x) = x^2 - 4x + 5$.



Need to find where the two curves meet in order to determine the limits for our integration.

So set $f(x) = g(x)$ and solve for x.

$$-x^2 + 4x - 1 = x^2 - 4x + 5$$

$$-2x^2 + 8x - 6 = 0$$

$$2x^2 - 8x + 6 = 0$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

$x = 1, 3$ So $a = 1$, $b = 3$.

Easy to see that f is the larger fn, but we can guarantee this by finding

$$f(2) = -2^2 + 4 \cdot 2 - 1 = 3$$

$$g(2) = 2^2 - 4 \cdot 2 + 5 = 1.$$

This shows $f(x) \geq g(x)$ on all of $[1, 3]$.

Then

$$A = \int_1^3 (f(x) - g(x))\,dx$$

$$= \int_1^3 (-2x^2 + 8x - 6)\,dx$$

(already found $f-g$ earlier).

$$= 2.667$$ (calculator).

Comparison of Integrals

Suppose we have two cts. fns $f(x)$ & $g(x)$ which are cts. on $[a,b]$. If $f(x) \le g(x)$ on $[a,b]$, then every term in the left-hand sum

$$\sum_{i=0}^{n-1} f(x_i)\Delta x$$

for $\int_a^b f(x)dx$ is $\le$ each term in the l.h. sum

$$\sum_{i=0}^{n-1} g(x_i)\Delta x$$

for $\int_a^b g(x)$.

Taking limits, we get.

Theorem 5.4 Comparison of Definite Integrals

Let f, g be cts on $[a, b]$.

1. If $m \leq f(x) \leq M$ on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

2. If $f(x) \leq g(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Ex. $\sin(x^2) \leq 1$ on $[0, \sqrt{\pi}]$, so

$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx \leq \int_0^{\sqrt{\pi}} 1\,dx = \sqrt{\pi}.$$