The Fundamental Theorem of Calculus

(First Part)

Earlier, we saw that for an object moving with displ. $s(t)$ & vel. $v(t)$, we had

$$\text{Change in displacement between } t=a \text{ \& } t=b = s(b) - s(a) = \int_a^b v(t)\,dt.$$

Since $s'(t) = v(t)$, we can write this as

$$s(b) - s(a) = \int_a^b s'(t)\,dt.$$

More generally, we have

Theorem 5.1 Fundamental Theorem of Calculus

If f is cts. on $[a,b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

or

$$\int_a^b F'(t)\,dt = F(b) - F(a)$$, the total change in $F(t)$ between $t=a$ and $t=b$.

$F(t)$ is called an antiderivative of $f(t)$.

To understand this, think of $f(t) = F'(t)$ as the rate of change of the quantity $F(t)$.
To calculate the total change in $F(t)$ between $t=a$ & $t=b$, divide $[a,b]$ into n equal subintervals of length $\Delta t = \frac{b-a}{n}$.
For each small subinterval, estimate the change in $F(t)$, called ΔF and add these changes together.

In each subinterval, we assume the rate of change of $F(t)$ is approx. constant, so that we can say

$$\Delta F \approx \text{Rate of change of } F \times \text{Time elapsed.}$$

For the first subinterval from t_0 to t_1, the rate of change is approx $F'(t_0)$, so

$$\Delta F \approx F'(t_0)\,\Delta t = f(t_0)\,\Delta t.$$

Similarly, for the second interval

$$\Delta F \approx F'(t_1)\,\Delta t,$$

etc.

Summing up over all the subintervals, we get

$$\text{Total change in } F \text{ between } t=a \text{ \& } t=b = \sum_{i=0}^{n-1} \Delta F \approx \sum_{i=0}^{n-1} F'(t_i)\,\Delta t.$$

and as $n \to \infty$, the sum on the right converges to $\int_a^b F'(t)\,dt$ by the defn of the definite integral.

Ex. Calculate

$$\int_0^1 2x\,dx.$$

Let $F(x) = x^2$. Then $F'(x) = 2x$ and so by FTOC

$$\int_0^1 2x\,dx = F(1) - F(0) = 1^2 - 0^2 = 1.$$

n.b. can also do this using area.

Ex 1. A bacteria colony initially has 14×10^6 bacteria. Suppose that the pop. incr. as a fn of time t in hours with rate 2^t million bacteria/hr.

a) Give a definite integral which represents the total change in the pop. during the time from $t=0$ to $t=2$.

$$\int_0^2 2^t \, dt$$

b) Find the pop. at $t=2$.

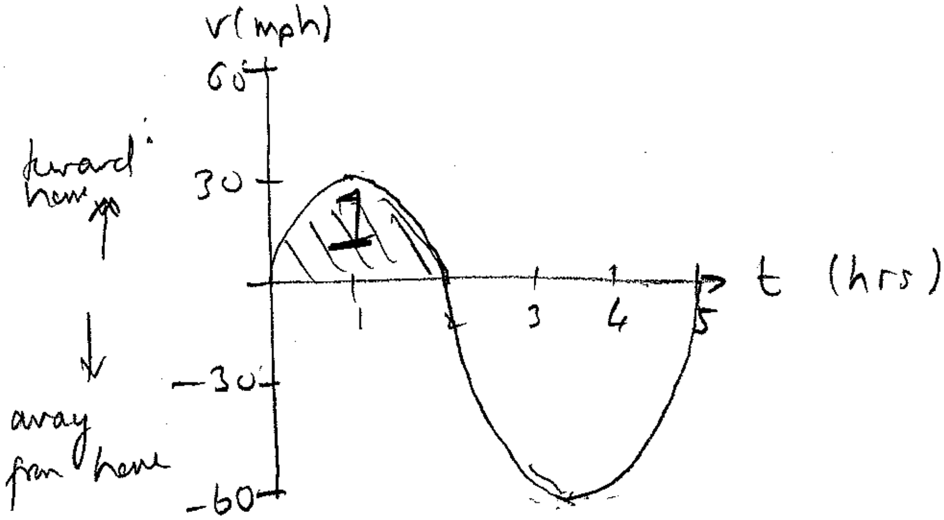
Using a calculator

$$\int_0^2 2^t \, dt = 4.328.$$

Started with 1.4×10^7 at $t=0$ and between $t=0$ & $t=2$, pop incr by 4.328×10^6.

Hence the pop at $t=2$ hrs is 18.328 million bacteria.

Ex3. Man starts 50 mi away from home & takes a trip in his car. Moves on a straight line & his home is on that line. Vel. is given in the following graph

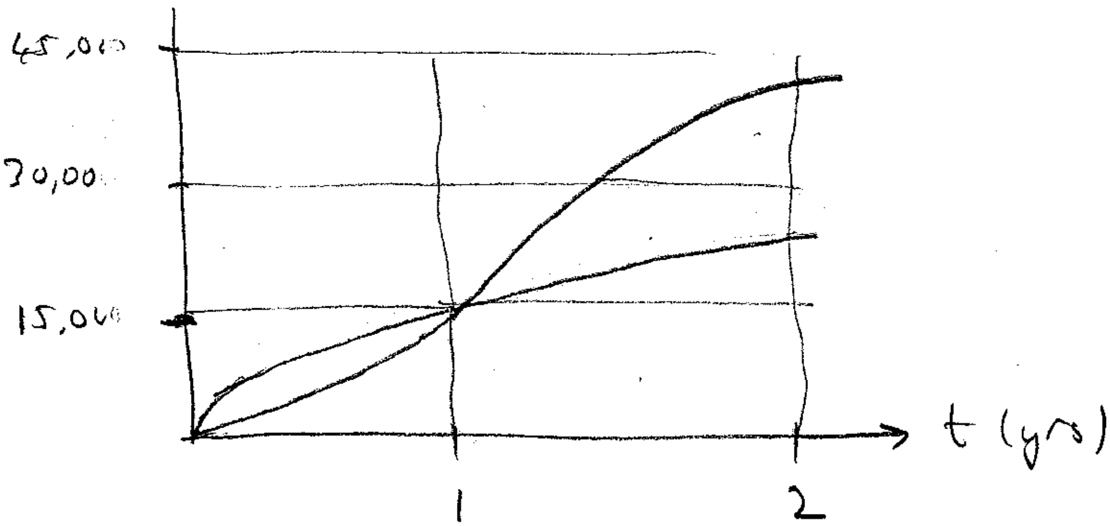


a) Closest to home after about 2 hr.

Dist travelled is area of shaded region 1 above axis. Each square has area 30 mph × 1 hr = 30 mi, so he is about 30 mi closer than he started out, ie about 20 mi from home.

b) Farthest away?

Ex 4. The rate of growth of the pops. of two species of plants (measured in new plants / yr) are as below



a) Which pop. is larger after 1 yr?
2 yrs?

b) How much does the pop. of species 1 incr. during the first two yrs.

~2.25 grid squares

1 grid sq = 15,000 plants/yr × 1 yr
= 15,000 plants.

~ 2.25 × 15,000 = 3,375 plants.

The Definite Integrals as an Average

Average of n numbers $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

- just sum and divide by n.

What about the average value of a continuously varying quantity such as temperature?

For example, if we want to get the av. temp. over a 24 hr. period, we can take measurements at n equally spaced time intervals Δt. Then

$$\text{Av. temp} \approx \frac{f(t_1) + \cdots + f(t_n)}{n}$$

The larger n, the better the approximation.

Can rewrite this as a Riemann sum if we remember that $\Delta t = \frac{24}{n}$, so

$n = \frac{24}{\Delta t}$.

Av temp $\approx \dfrac{f(t_1) + \cdots + f(t_n)}{24/\Delta t}$

$$= \frac{f(t_1)\Delta t + \cdots + f(t_n)\Delta t}{24}$$

$$= \frac{1}{24}\left(f(t_1)\Delta t + \cdots + f(t_n)\Delta t\right)$$

$$= \frac{1}{24} \sum_{i=1}^{n} f(t_i)\Delta t.$$

as we let $n \to \infty$, this Riemann sum tends to

$$\frac{1}{24}\int_0^{24} f(t)\,dt$$

and we call this the av. temp.

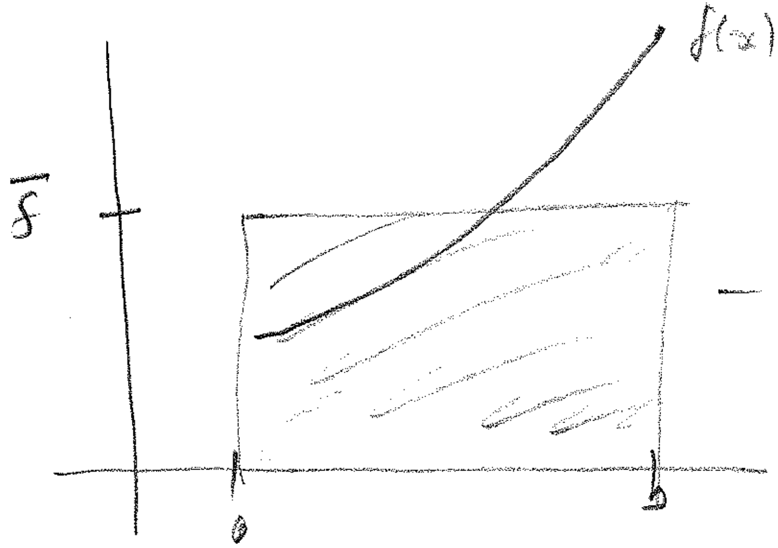
In general, for f cts on $[a,b]$, we define the average value $\bar{f}$ of f on $[a,b]$ to be

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

So

$$\bar{f}\,(b-a) = \int_a^b f(x)\,dx.$$

Thus a rectangle of height $\bar{f}$ over $[a,b]$ has the same (signed) area as that under the graph of f.



— area of rectangle = area under graph of f.

Ex2. Suppose $C(t)$ repr. the cost / day of heating your home in \$/day where time t is measured in days, since 1/1/06.

Interpret;

$$\int_0^{90} C(t)\,dt, \quad \frac{1}{90-0}\int_0^{90} C(t)\,dt.$$

Units are \$/day × day = \$.

$\int$ repr. total cost of heating your home for the first 90 days of 2006, i.e. for Jan, Feb, Mar '06.

$$\frac{1}{90-0}\int_0^{90} C(t)\,dt$$ is the average

cost per day of heating your house for the first 90 days of 2006.

Theorems About Definite Integrals

Theorem 5.2 Properties of Limits of Integration

If a, b, c are real numbers and f is a cts fn, then

1. $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

2. $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Idea for 1

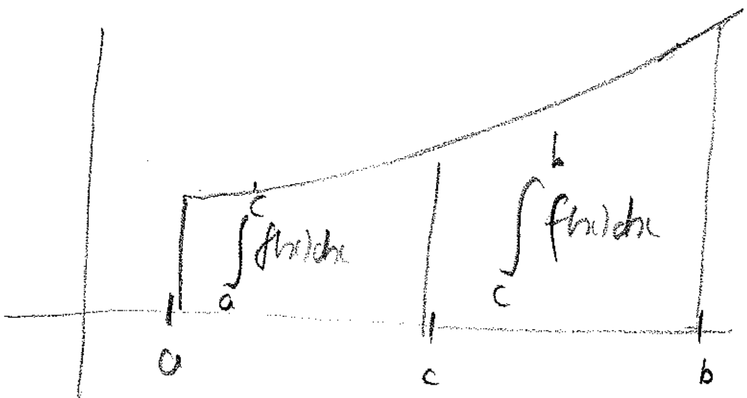
Subintervals for $\int_b^a f(x)\,dx$ have width $\frac{a-b}{n}$

" " " $\int_a^b f(x)\,dx$ " " $\frac{b-a}{n} = -\frac{(a-b)}{n}$.

Everything else involved in obtaining the two integrals is the same.

Idea for I2

Case $a < c < b$ (other cases can be obtained from this one and I.)



combining gives $\int_a^b f(x)\,dx$.

Even and Odd Functions

If $f(x)$ is <u>even</u> $(f(-x) = f(x))$, then

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx, \quad \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

If $f(x)$ is <u>odd</u> $(f(-x) = -f(x))$

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx, \quad \int_{-a}^{a} f(x)\,dx = 0.$$

Ex. Given $\int_0^{\pi} \sin x\,dx = 2$, find.

a) $\int_{-\pi}^{\pi} \sin x\,dx$, b) $\int_{-\pi}^{\pi} |\sin x|\,dx$.

a)
Since $\sin x$ is odd, $\int_{-\pi}^{\pi} \sin x\,dx = 0$.

b) Since $|\sin x|$ is even,

$$\int_{-\pi}^{\pi} |\sin x|\,dx = 2\int_0^{\pi} \sin x\,dx = 2 \times 2 = 4.$$

Ex 1. Suppose $\int_0^{1.25} \cos(x^2)\,dx = 0.98$

and $\int_0^{1} \cos(x^2)\,dx = 0.9.$

Find

a) $\int_1^{1.25} \cos(x^2)\,dx$ $\qquad \int_0^{1.25} \cos(x^2)\,dx + \int_{1.25}^{1} \cos(x^2)\,dx$ by 2.

$$= \int_0^{1.25} \cos(x^2)\,dx - \int_1^{1.25} \cos(x^2)\,dx \quad \text{by } 1$$

$$= .98 - .9 = 0.08.$$

b) $\int_{-1}^{1} \cos(x^2)\,dx = 2\int_0^{1} \cos(x^2)\,dx$ as $\cos(x^2)$ is <u>even</u>

$$= 2 \times .9 = 1.8.$$

c) $\int_{1.25}^{-1} \cos(x^2)\,dx = -\int_{-1}^{1.25} \cos(x^2)\,dx$ by 1.

$$= -\left(\int_{-1}^{1} \cos(x^2)\,dx + \int_{1}^{1.25} \cos(x^2)\,dx\right)$$

by 2.

$$= -\left(1{\cdot}8 + {\cdot}08\right) \quad \text{by} \quad b) \ \& \ a).$$

$$= -1{\cdot}88.$$