Sigma notation.

For n numbers $a_1, a_2, \ldots, a_{n-1}, a_n$,
by $\sum_{i=1}^{n} a_i$ we mean $a_1 + a_2 + \cdots + a_n$.

ie

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

e.g.

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

Suppose now we have a fn f which is cts on $[a,b]$.

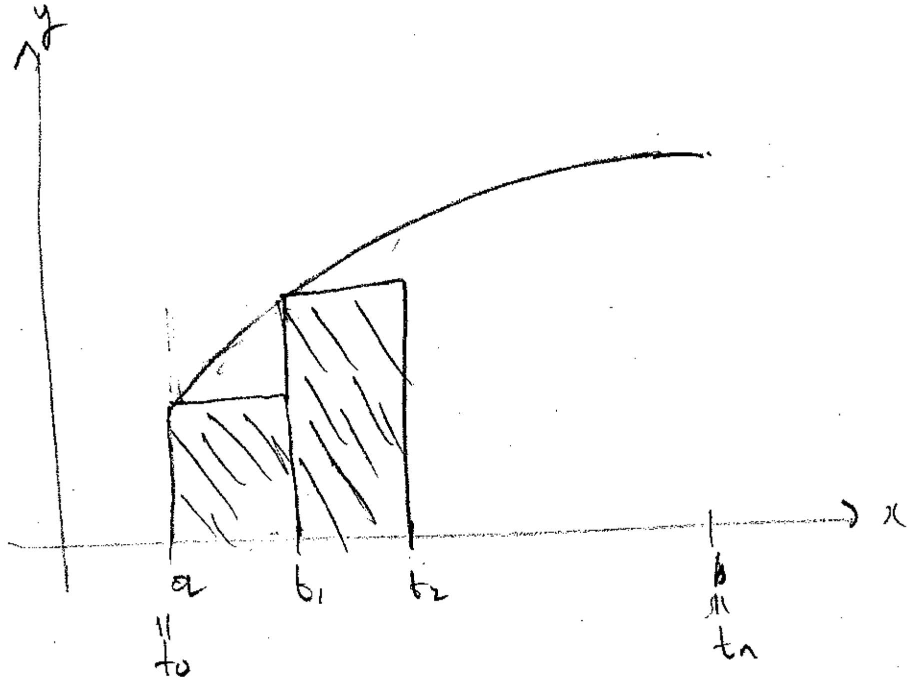
We divide $[a,b]$ into n equal subintervals, each of which has width

$$\Delta t = \frac{b-a}{n}.$$

Let $t_0, t_1, \_\_\_\_, t_n$ be the endpts.
of the subdivisions.

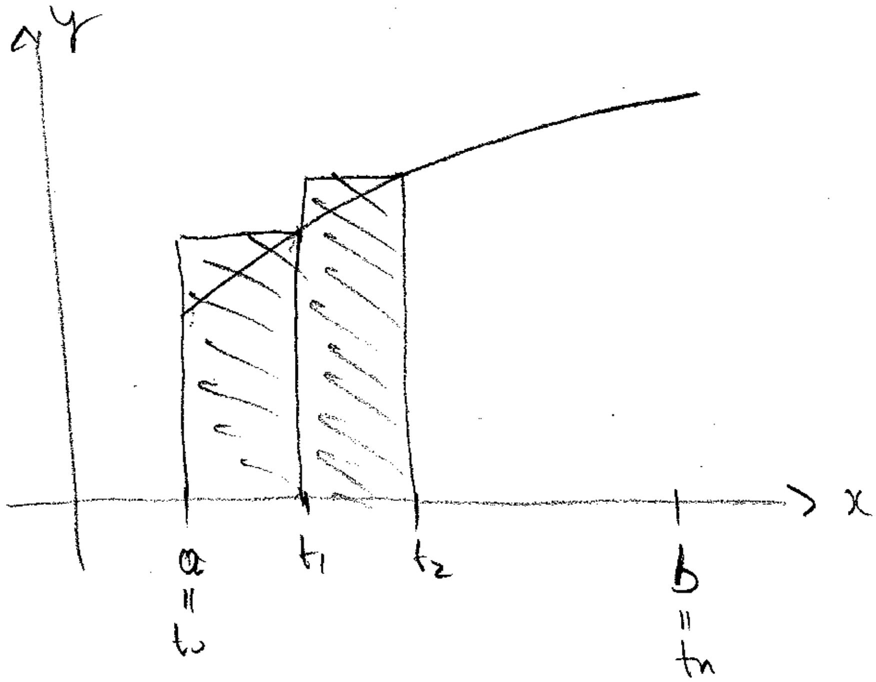
e.g the first interval is $[t_0, t_1]$ & is in fact
$[a, a+\Delta t] = [a, a+\frac{(b-a)}{n}]$.

the second interval is $[t_1, t_2]$ & is
$[a+\frac{(b-a)}{n}, a+2\frac{(b-a)}{n}]$.



Form the left-hand sum

$$f(t_0)\Delta t + f(t_1)\Delta t + \_\_\_ + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t$$



and the right-hand sum

$$f(t_1)\Delta t + f(t_2)\Delta t + \text{———} + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t.$$

If f is either incr or decr., then

$$\left| \begin{matrix}\text{Difference between} \\ \text{upper and lower estimates}\end{matrix} \right| = \left| \begin{matrix}\text{Difference} \\ \text{between} \\ f(b)\ \&\ f(a)\end{matrix} \right| \cdot \Delta t$$

$$= |f(b) - f(a)| \cdot \Delta t$$

$$= |f(b) - f(a)| \cdot \frac{(b-a)}{n}.$$

Note that this $\to 0$ as $n \to \infty$.

This motivates the following very important defn.

Definite Integrals.

Suppose f cts on $[a, b]$. The definite integral of f from a to b, written

$$\int_a^b f(t)\,dt$$

is the limit of the left-hand or right-hand sums with n subdivisions of $[a, b]$ as n gets arbitrarily large. In other words

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Left-hand sum}) = \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} f(t_i)\,\Delta t \right)$$

and

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Right-hand sum}) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(t_i)\,\Delta t \right)$$

Each of these sums is called a <u>Riemann Sum</u>, f is called an <u>integrand</u> & a, b are the <u>limits of integration</u>.

<u>Facts</u>. For f cts on $[a,b]$, these limits always exist and are the same. Hence $\int_a^b f(t)dt$ exists in this case.

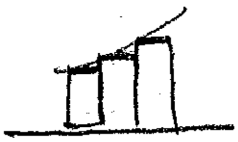
We can also consider more general Riemann sums where we take a subdivision $a = t_0 < t_1 < \ldots < t_n = b$ into subintervals whose size may or may not be equal. We then pick a point c_i in each interval $[t_{i-1}, t_i]$ and form the sum

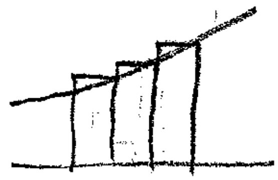
$$\sum_{i=1}^{n} f(c_i)\Delta t_i \quad \text{where} \quad \Delta t_i = t_i - t_{i-1}, \text{ the width of } [t_{i-1}, t_i].$$

For f cts on $[a,b]$, these sums must also converge to $\int_a^b f(t)dt$, provided the width of the <u>widest</u> subinterval tends to zero.

If f is incr., then

$$\text{l.h. sum} \leq \int_a^b f(t)\,dt \leq \text{r.h. sum}$$





If f is decr., then

$$\text{r.h. sum} \leq \int_a^b f(t)\,dt \leq \text{l.h. sum.}$$

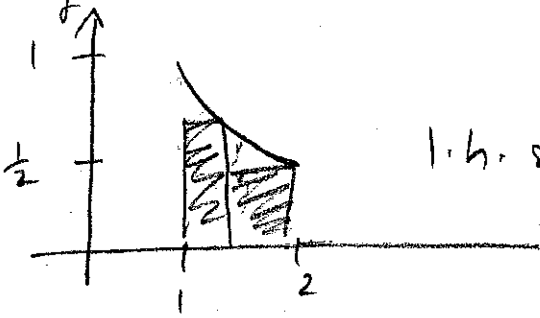
In either case

$$\left| \int_a^b f(t)\,dt - \text{l.h. sum} \right| < |f(b) - f(a)|\Delta t$$

$$\left| \int_a^b f(t)\,dt - \text{r.h. sum} \right| < |f(b) - f(a)|\Delta t.$$

Ex 1 Compute Riemann sums for

$\int_1^2 \frac{1}{t}\,dt$ with $n=2$, $n=10$, $n=250$.



l.h. sum.

Here $a=1$, $b=2$.

For $n=1$ $\Delta t = 1$.

l.h. sum is $f(1)\Delta t + f(1.5)\Delta t = 1(.5) + \frac{1}{1.5}\times .5 = .833\ldots$

r.h. sum is $f(1.5)\Delta t + f(2)\Delta t = \frac{1}{1.5}\times .5 + \frac{1}{2}\times .5 = .5833\ldots$

So $.5833 < \int_1^2 \left(\frac{1}{t}\right) dt < .833\ldots$

For $n=10$, $\Delta t = \frac{1}{10}$

l.h. sum is $f(1)\Delta t + f(1.1)\Delta t + \cdots + f(1.9)\Delta t$

$= 1\times .1 + \cdots + \frac{1}{1.9}\times .1 = .7188$

r.h. sum $f(1.1)\Delta t + \cdots + f(2)\Delta t = .6688$

Get

$$.6688 < \int_1^2 \frac{1}{t}\,dt < .7188$$

With $n = 250$, l.h. sum is $.6921$ & r.h. sum is $.6941$.

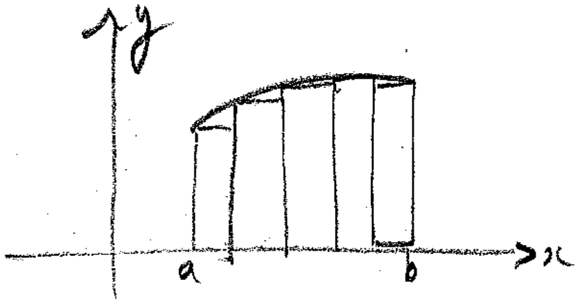
Hence

$$.6921 < \int_1^2 \frac{1}{t}\,dt < .6941.$$

Actually, $\int_1^2 \frac{1}{t}\,dt = \ln 2 = .6931\ldots$

The Definite Integral as an Area

If $f \geq 0$ on $[a,b]$, then the left- and right-hand sums approx. the area under the graph of f.

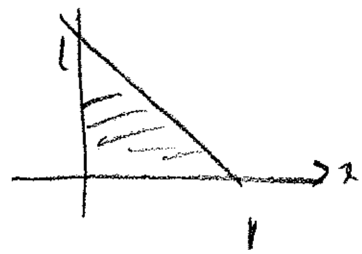


In this case, we define the area under the graph to be

$$\int_a^b f(x)\,dx.$$

Ex2.

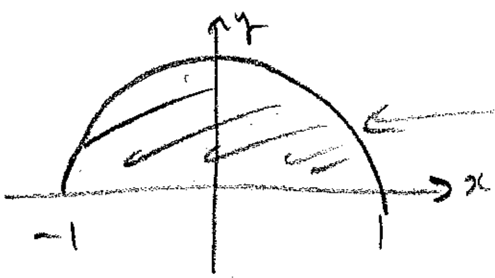
a) $$\int_0^1 (1-x)\,dx = \frac{1\times 1}{2} = \frac{1}{2}$$



as this is the area of a triangle of height 1 & base length 1

b) $$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{\pi \cdot 1^2}{2} = \frac{\pi}{2}$$

as this is the area of a semicircle of radius 1

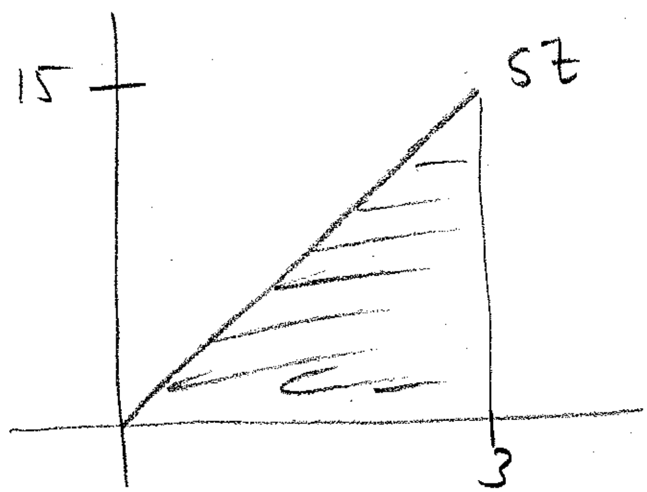


$x^2+y^2=1$

$y = +\sqrt{1-x^2}$

Ex 2 With time t in s, the vel. of a bicycle in ft/s. is given by $v(t) = 5t$. How far does the bicycle travel in 3 s?

Soln. vel. is linear



Dist. is area of a triangle of height 15 & base 3.

Hence

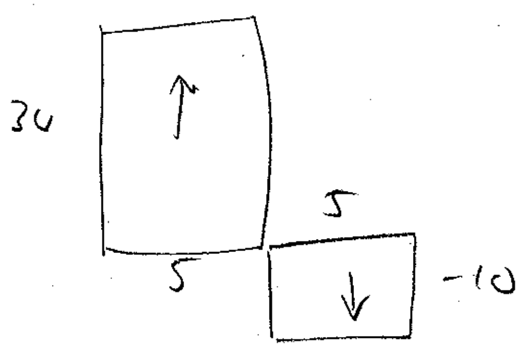
$$\text{Dist} = \frac{1}{2}\, 3 \times 15 = 22.5 \text{ ft.}$$

Ex. A particle moves along the y-axis with vel. 30 cm/s for 5 s and vel. -10 cm/s for the next 5 s.

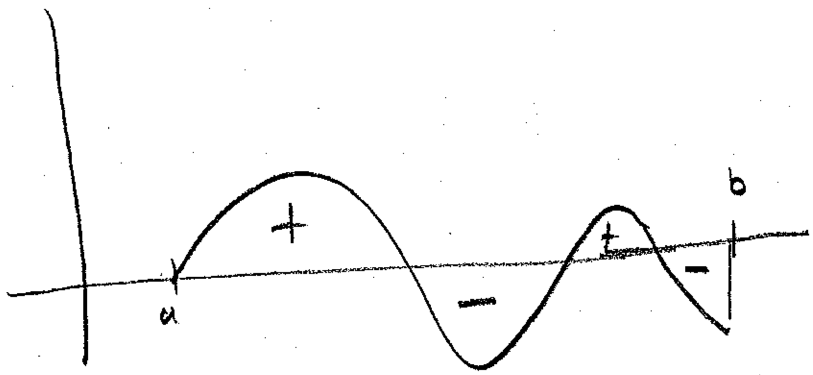
+ve vel represents upward motion and -ve vel. — downward motion.

What is represented by the sum

$$30 \cdot 5 + (-10) \cdot 5 \, ?$$

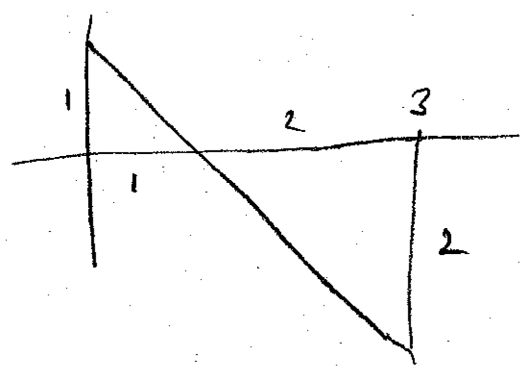


When $f(x)$ is not positive



$\int_a^b f(x)\,dx$ is the sum of areas above the x-axis counted positively, and areas below the x-axis counted negatively.

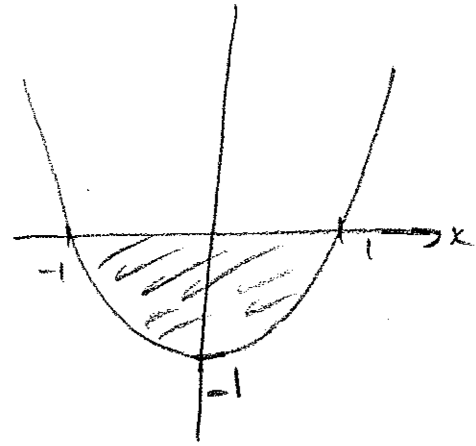
Ex3. a). $\int_1^3 (1-x)\,dx$



This is the area of the Δ above the axis — the area of the Δ below the axis.

Hence $$\int_1^3 (1-x)\,dx = \frac{1}{2}(1\times 1) - \frac{1}{2}(2\times 2) = \frac{1}{2} - 2 = -\frac{3}{2}.$$

b) $$\int_{-1}^{1} (x^2-1)\,dx.$$



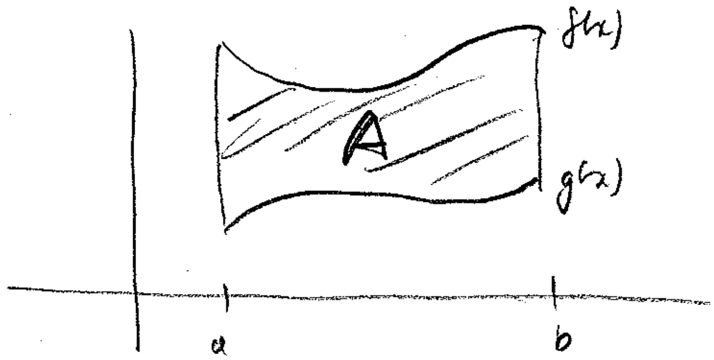
This is the area of the region below the axis for the parabola $y = x^2-1$ as shown.

It is $-4/3$. (see later).

Area between Two Curves

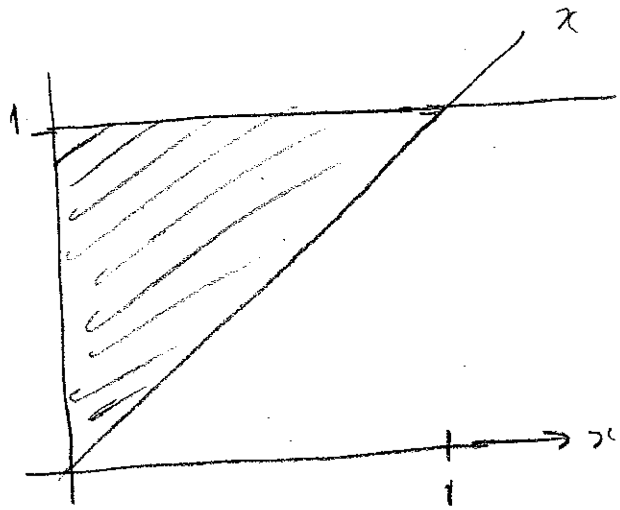
If f, g are cts. on $[a, b]$ and $g(x) \leq f(x)$, $\forall x \in [a,b]$, then the area between the graphs of f & g is given by

$$A = \int_a^b (f(x) - g(x))\,dx$$



Ex. Find the area between $f(x) = 1$ and $g(x) = x$ for $0 \leq x \leq 1$.

First note that $f(x) \geq g(x)$ on $[0, 1]$.



The region between the two curves is a Δ of height 1 & base 1. Hence the area between the two curves is

$$A = \frac{1}{2} 1 \times 1 = \frac{1}{2}.$$