§ 4.7 L'Hôpital's Rule.

Suppose we have two fns f & g which are diff at $x = a$ and for which $f(a) = g(a) = 0$.

Suppose also for now that $g'(a) = 0$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{g(a+h) - g(a)}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \frac{h}{g(a+h) - g(a)}$$

$h \neq 0$ for limit as $h \to 0$

$$= \lim_{h \to 0} \frac{\frac{f(a+h) - f(a)}{h}}{\frac{g(a+h) - g(a)}{h}}$$

$$= \frac{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}{\lim_{h \to 0} \frac{g(a+h) - g(a)}{h}}$$

$$= \frac{f'(a)}{g'(a)}$$

We can actually prove a more general version where we do away with the requirement that $g'(a) \neq 0$.

<u>l'Hôpital's Rule (First Version)</u>

If f and g are diff at $x = a$ and $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Ex1.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$\sin x$ & x are both diff at $x=0$

and $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(x) = 1.$ $(\neq 0)$

Hence, by l'Hôpital.

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos 0}{1} = 1$$

(as we already knew!).

Ex2.

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}.$$

$1-\cos x$, x^2 are both diff at $x=0$
& both have value 0 & 0.

$\frac{d}{dx}(1-\cos x) = -(-\sin x) = \sin x$

$\frac{d}{dx}(x^2) = 2x$

By l'Hôpital,

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x}$$

r.h. limit is again of $\frac{0}{0}$ type.
Can either do l'Hôpital again or use Ex1. to get

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2}.$$

This illustrates an important fact when dealing with l'Hôpital's rule.

<u>L'Hôpital's rule is a handy trick for doing limits, but in order to get it to work, one often has to apply it more than once.</u>

L'Hôpitals Rule for Limits Involving Infinity (Second, Third and Fourth forms).

Provided f and g are diff. :

- When $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$

or

- When $a = \infty$ (or $-\infty$) and

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ or $\lim_{x \to \infty} f(x) = \pm\infty$ and $\lim_{x \to \infty} g(x) = \pm\infty$,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(where a may be $\pm\infty$), provided the limit on the r.h.s. exists.

Ex3

$$\lim_{x\to\infty} \frac{5x + e^{-x}}{7x}$$

$5x + e^{-x} \to \infty$
as $x \to \infty$
and $7x \to \infty$

and both fns are diff, so by l'Hôpital.

$$\lim_{x\to\infty} \frac{5x + e^{-x}}{7x} = \lim_{x\to\infty} \frac{5 - e^{-x}}{7} = 5/7.$$

Ex 4. $\lim_{x\to\infty} x^2 e^{-x}$

Write this as

$$\lim_{x\to\infty} \frac{x^2}{e^x}$$

Both $x^2, e^x \to \infty$ as $x \to \infty$ and are diff.

Hence $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{2x}{e^x}$

and by using the same argument again and l'Hôpital, this is

$$= \lim_{x \to \infty} \frac{2}{e^x} = 0.$$

<u>Ex 5</u>. $\displaystyle \lim_{x \to \infty} \left(1+\frac{a}{x}\right)^x$

Since $\left(1+\frac{a}{x}\right)^x \longrightarrow 1^\infty$ as $x \to \infty$

and 1^∞ makes no sense, we write

$$y = \left(1+\frac{a}{x}\right)^x$$

and find the limit of $\ln y$.

$$\ln y = \ln\left(1+\frac{a}{x}\right)^x = x \ln\left(1+\frac{a}{x}\right).$$

$$= \frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{x}}.$$

$\ln(1+\frac{a}{x})$, $\frac{1}{x}$ are both diff & go to 0 as $x \to \infty$, so by l'Hôpital,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+a/x)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+a/x} \cdot -\frac{a}{x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{+a}{1+\frac{a}{x}} \quad (x \neq 0 \text{ for limit as } x\to\infty!)$$

$$= a.$$

Since $\lim_{x\to\infty} \ln y = a$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^a$$

Hence

$$\boxed{\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a}$$

(for any real number a).

<u>Domination</u>

Say a fn g <u>dominates</u> a fn f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

L'Hôpital is useful for checking this.

Ex 6. Check $x^{\frac{1}{2}}$ dominates $\ln x$

as $x \to \infty$.

Apply l'Hôpital to $\dfrac{\ln x}{x^{\frac{1}{2}}}$.

$$\lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{2}}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{2}x^{-\frac{1}{2}} \cdot x}$$

$$= \lim_{x\to\infty} \frac{2}{x^{\frac{1}{2}}}$$

$$= 0.$$

Ex. Check that any exp. fn of the form e^{kx} with $k>0$ dominates any fixed power of the form Ax^p where $p>0$, as $x \to \infty$.

Apply l'Hôpital repeatedly to $\frac{Ax^p}{e^{kx}}$.

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = \lim_{x\to\infty} \frac{Apx^{p-1}}{ke^{kx}} = \lim_{x\to\infty} \frac{Ap(p-1)x^{p-2}}{k^2e^{kx}} = \cdots$$

Keep doing this until the power of x is no longer positive. The limit on the top is then finite while that on the bottom is still ifinity.

Hence

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = 0$$

and e^{kx} dominates Ax^p.