§ 4.6 Rates and Related Rates

Ex1. A spherical snowball is melting. Its radius decr. at a const rate of of 2cm/min from an initial value of 70 cm. How fast is the vol. decr after half an hour?

Radius r starts at 70 cm & decr at 2cm/min.
At t min since the start

$$r = 70 - 2t \text{ cm.}$$

Volume of snowball is

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3.$$

To get $\frac{dV}{dt}$, need V as a fn of t. So we subst for r as a fn of t in V.

$$V(t) = \frac{4}{3}\pi(70-2t)^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3(70-2t)^2 \cdot -2 \qquad \text{chain rule.}$$

$$= -8\pi(70-2t)^2.$$

When $t = 30$ min

$$\frac{d}{dt}V = -8\pi(70-2\cdot 30)^2$$

$$= -800\pi \ \text{cm}^3/\text{min}.$$

Note $\frac{dV}{dt} < 0$ which reflects the fact that the snowball is loosing volume.

Ex2. A skydiver of mass m jumps from a plane at time $t=0$. The dist s he has fallen is given by

$$s(t) = \frac{m^2 g}{k^2}\left(\frac{kt}{m} + e^{-kt/m} - 1\right)$$

for some +ve const. k.

a) Find $s(0)$ & $s''(0)$ & interpret in terms of the skydiver.

b). Relate the units of $s'(t)$ and $s''(t)$ to the units of t and $s(t)$.

a) Diff using chain rule gives:

$$s'(t) = \frac{m^2 g}{k^2}\left(\frac{k}{m} + e^{-kt/m}\left(-\frac{k}{m}\right)\right)$$

$$= \frac{mg}{k}\left(1 - e^{-kt/m}\right)$$

$$s''(t) = \frac{mg}{k} \cdot \left(-e^{-kt/m}\right)\left(-\frac{k}{m}\right) = g e^{-kt/m}.$$

Since $e^{-k \cdot 0/m} = 1$, evaluating at $t=0$ gives.

$s'(0) = \frac{mg}{k}(1-1) = 0$, $s''(0) = g$.

Starts with 0 initial vel. & initial accel g.

b) units of $s'(t)$ = $\frac{\text{units of } s}{\text{units of } t}$ = $\frac{\text{dist}}{\text{time}}$

(eg m/s).

" " s'' " " $\frac{s'}{t}$ = $\frac{\frac{\text{dist}}{\text{time}}}{\text{time}}$

$= \frac{\text{dist}}{(\text{time})^2}$ (eg. m/s^2).

Steps for Solving Related Rates Problems

0. Read the question, stupid!

1. Draw a picture if needed.

2. What is the quantity whose rate we want?

   What is the quantity whose rate we are given?

3. Find a formula for the quantity whose rate we want.
   (mensuration, trigonometry, Pythagoras).

4. Obtain a relationship between the quantity whose rate we want and the quantity whose rate we are given.

   Requires information from the question and/or mensuration, trig, Pyth. etc.

5. Differentiate (usually involves the chain rule) or implicit differentiation

6. Substitute appropriate values to get the answer.

Ex 3

A sph. snowball melts in such a way that at the instant at which its radius is 20cm, it's rad. is decr. at 3 cm/min. At what rate is the vol of the ball of snow changing at that instant?

0! 1 (ok).

2. Given $\frac{dr}{dt} = -3$ cm/min at $r = 20$cm.

Want $\frac{dV}{dt}$ at $r = 20$ cm.

3. $V = \frac{4}{3}\pi r^3$

4. Can't actually get V as a fn of t, but we don't need to here.

5. $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$ — chain rule

$= 4\pi r^2 \frac{dr}{dt}$

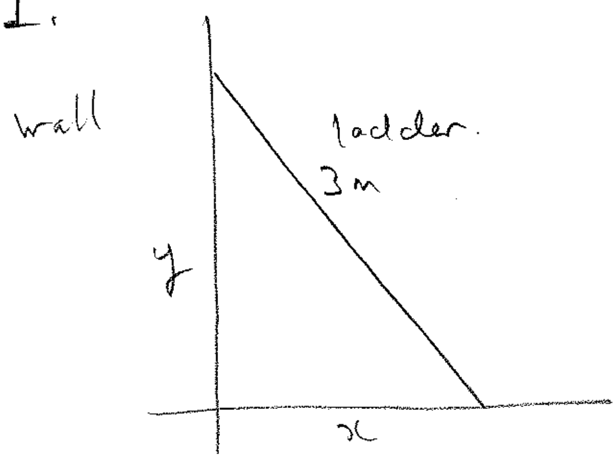
6. When $r = 20$, $\frac{dr}{dt} = -3$ cm/min and

$\frac{dV}{dt} = 4\pi \cdot (20)^2 \cdot -3 = -4800$ cm^3/min.

Ex 4. The Ladder Problem

a) A 3m ladder stands against a high wall. The foot of the ladder is moving outward at a speed of .1 m/s when the foot is 1 m from the wall. At that moment, how fast is the top of the ladder falling?.

1. O!



2. Want $\frac{dy}{dt}$ when $x = 1$

Given $\frac{dx}{dt} = .1$ m/s when $x = 1$.

3/4. By Pythagoras

$$x^2+y^2 = 3^2.$$

5. Diff impl wrt t.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 1m$, $\frac{dy}{dt} = 0.1 m/s$ &

$y = \sqrt{3^2-1^2} = \sqrt{8}\ m$ so

$$\frac{dy}{dt} = -\frac{1}{\sqrt{8}} \times .1 = -0.035 m/s.$$

b) If the foot of the ladder moves out at a const speed, how does the speed at which the top falls change as the foot moves out?

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

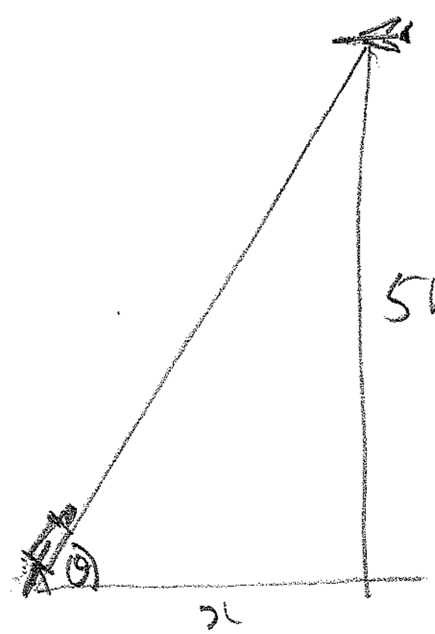
So if $\frac{dx}{dt}$ is const, as x incr,
y decr & so

$\frac{dy}{dt}$ gets more negative as
x incr. ie the top falls faster
as the foot of the ladder moves
further out.

Ex 5 A plane flying at 450 km/hr at a const altitude of 5 km is approaching a camera on the ground. Let θ be the angle of elevation above the (flat) ground at which the camera is pointed. When $\theta = \frac{\pi}{3}$, how fast does the camera have to tilt in order to keep the plane in view?

0.!

1.



Let x be the horiz. dist along the ground from the plane to the camera.

2. Want $\frac{d\theta}{dt}$, when $\theta = \frac{\pi}{3}$.

Have $\frac{dx}{dt} = -450$ km/hr

(why − sign?)

3/4. $\tan\theta = 5/x$

5. Diff impl. wrt t.

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{5}{x^2}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \cos^2\theta \cdot -\frac{5}{x^2}\frac{dx}{dt}$$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$,

$\frac{5}{x} = \tan\frac{\pi}{3} = \sqrt{3}$, so

$x = 5/\sqrt{3}$

and $\frac{dx}{dt} = -450$ km/hr.

So

$$\frac{d\theta}{dt} = \frac{1}{4} \cdot -5/_{25/3} \cdot -450$$

$$= \frac{1}{4} \cdot -\frac{3}{5} \cdot -450$$

$$= \frac{270}{4}$$

$= 67 \cdot 5$ rad / hr.