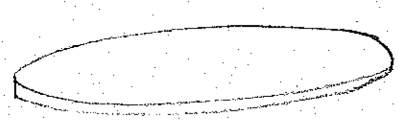
§ 4.6 Optimization and Modelling

Virgil's Aeneid.

Ex1. What are the dimensions of an aluminium can that holds 40 in^3 of juice & which uses the least amount of Aluminium. Assume the can is cylindrical and capped at both ends.

Thinking about the problem in general terms

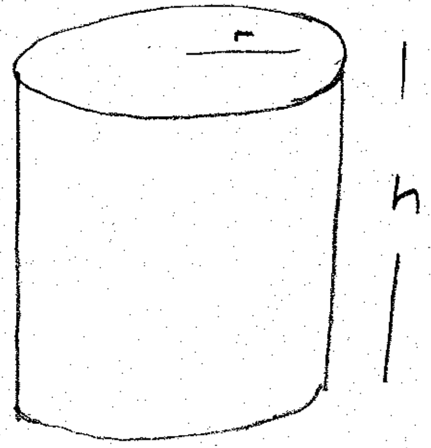


very flat can
- inefficient

very thin can.
(probably) also inefficient.

Since the extremes seem to be inefficient, suggests that the best proportions for our can lie somewhere in the middle.

Let's make a picture.

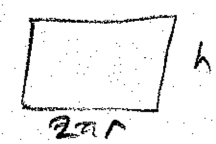


Q. What is the quantity to be optimized?

A. The area of aluminium used.

Can has one curved side and two flat circular sides.

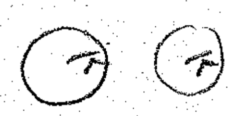
The area of the curved side is



$2\pi r h$ (imagine cutting along the seam and rolling flat).

The area of the two flat sides is

$\pi r^2 + \pi r^2 = 2\pi r^2$



Hence the total area to be minimized is

$A = 2\pi r h + \pi r^2.$

We now come to what is (usually) the hardest part - eliminating the unwanted variable so that A is a fn of a single variable (r or h).

For this we need a relationship between r & h. Can we find one?

Well the volume must be 40 in^3.

Since the can is a cylinder, we have

$$V = \pi r^2 h = 40.$$

Hence $h = \dfrac{40}{\pi r^2}$.

Now subst this in our formula for A.

$$A = 2\pi r \cdot \frac{40}{\pi r^2} + 2\pi r^2$$

$$= \frac{80}{r} + 2\pi r^2.$$

Q. What is the correct range for r?

A. $0 < r < \infty$.

So we want to minimise

$$A(r) = \frac{80}{r} + 2\pi r^2 \quad \text{on} \quad (0, \infty).$$

Crit pts.

$$A'(r) = -\frac{80}{r^2} + 4\pi r = 0$$

$$\Rightarrow \quad 4\pi r = \frac{80}{r^2}$$

so

$$4\pi r^3 = 80$$

$$r^3 = \frac{20}{\pi}$$

$$r = \sqrt[3]{\frac{20}{\pi}} \simeq 1.85 \text{ m}.$$

Now

$$A''(r) = \frac{160}{r^3} + 4\pi$$

and for $r = \sqrt[3]{\frac{20}{\pi}}$, we have

$$A'(r) = \frac{160}{\left(\sqrt[3]{\frac{20}{\pi}}\right)^3} + 4\pi$$

$$= \frac{160}{\frac{20}{\pi}} + 4\pi$$

$$= 8\pi + 4\pi$$

$$= 12\pi > 0.$$

Hence by the 2nd deriv test.

$r = \sqrt[3]{\frac{20}{\pi}}$ is a local min.

Now we look at the endpts.

As $r \to 0$ $A(r) \to \infty$ and
as $r \to \infty$ $A(r) \to \infty$ also.

Hence the local min at $\sqrt[3]{\frac{20}{\pi}}$ is also a global min.

Here $h = \frac{40}{\pi r^2} = \frac{40}{\pi\left(\sqrt[3]{\frac{20}{\pi}}\right)^2} = \frac{40}{\pi\left(\frac{20}{\pi}\right)^{2/3}}$

$$= 2 \cdot \frac{20}{\pi \cdot \frac{20^{2/3}}{\pi^{2/3}}} = 2\left(\frac{20}{\pi}\right)^{\frac{1}{3}}$$

and

$$A = 2\pi r h + 2\pi r^2$$

$$= 2\pi \sqrt[3]{\frac{20}{\pi}} \cdot 2\sqrt[3]{\frac{20}{\pi}} + 2\pi\left(\sqrt[3]{\frac{20}{\pi}}\right)^2$$

$$= 6\pi\left(\frac{20}{\pi}\right)^{2/3} \approx 64.17 \text{ in}^2.$$

Note that in the end

$$h = 2r$$

which makes our can as close to a sphere as possible.

Since the sphere encloses the max. volume for the min surface area (e.g. soap bubbles), this answer seems reasonable.

General Procedure for Solving Optimization Problems - Seven Steps to Heaven

0. Read the question stupid!

1. Draw a (big) picture. Think about the problem in general terms first.

2. Determine the quantity to be optimized and find a formula for it - label your diagram accordingly.

3. Eliminate the unwanted variables. The information for this comes either from (rereading) the question or from the diagram. Commonly used things here are mensuration formulae, Pythagoras, trig (defn of sin, cos, tan).

4. Now that you have the quantity to be optimized, determine the interval on which it is to be optimized.

5. Find the global max/min as appropriate as explained in the last section.

6. Substitute to get your answer

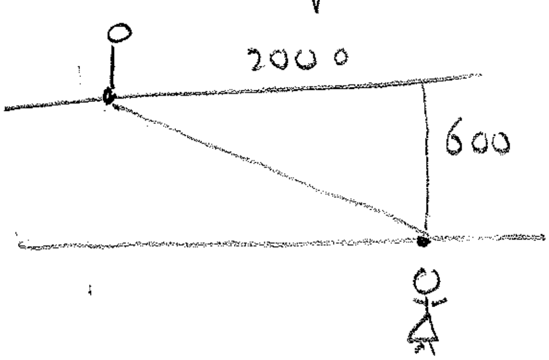
7. Check with your intuition to see if your answer is reasonable.

Ex2. Alina wants to get to the bus as quickly as possible. The bus stop is across a grassy park 2000 ft west and 600 ft north of her starting position. Alina can walk at 6 ft/sec on the sidewalk and 4 ft/sec on the grass. What path will get her to the bus stop as fast as possible.

O!

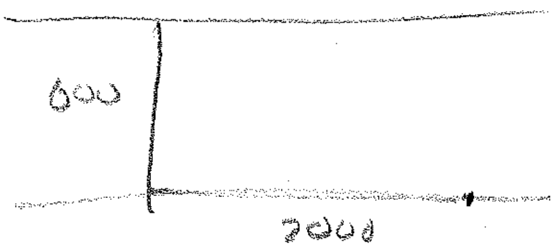
1. $v = s/t$, so $t = s/v$.

If she goes straight across the grass,



she traverses $\sqrt{2000^2+600^2} \approx 2100$ ft at 4 ft/s which takes about 525 s.

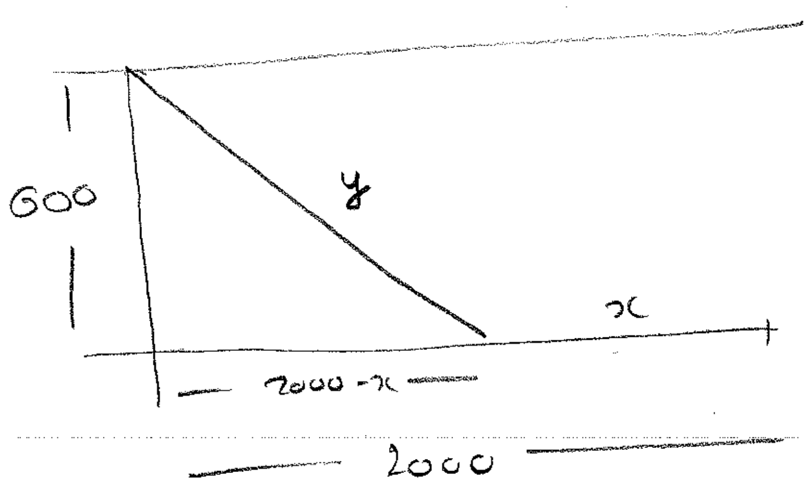
If she walks 2000ft along the path and then the minimum amount of 600ft on grass,



she takes $\frac{2000}{6} + \frac{600}{4} \approx 483s.$

Reasonable to assume the optimal path will consist of 2 straight line segments and be somewhere in between these two extremes.

2.



Total time is

$$T = \frac{x}{6} + \frac{y}{4}$$

3. Eliminate y.

<u>From Pythageras</u>

$$y^2 = (2000-x)^2 + 600^2$$

$$y = \sqrt{(2000-x)^2 + 600^2}$$

So $$T(x) = \frac{x}{6} + \frac{\sqrt{(2000-x)+600^2}}{4}$$

4. <u>Interval</u> $0 \leq x \leq 2000$ (why?).
<u>closed</u>.

5. <u>Crit pts</u>.

$$\frac{dy}{dx} = \frac{1}{6} + \frac{1}{4}\cdot\frac{1}{2}\left((2000-x)^2+600\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left((2000-x)^2+600^2\right)$$

$$= \frac{1}{6} + \frac{1}{4}\cdot\frac{1}{2}\left((2000-x)^2+600\right)^{-\frac{1}{2}} \cdot 2(2000-x)\cdot(-1)$$

$$= \frac{1}{6} - \frac{1}{4}\frac{2000-x}{\sqrt{(2000-x)^2+600^2}}$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{2000-x}{4\sqrt{(2000-x)^2+600^2}} = \frac{1}{6}$$

$$6(2000-x) = 4\sqrt{(2000-x)^2+600^2}$$

$$3(2000-x) = 2\sqrt{(2000-x)^2+600^2}$$

$$9(2000-x)^2 = 4(2000-x)^2 + 4\cdot 600^2$$

$$5(2000-x)^2 = 4\cdot 600^2$$

$$2000-x = \sqrt{\tfrac{4}{5}}\,600$$

$$x = 2000 - \sqrt{\tfrac{4}{5}}\cdot 600$$

$$\approx 1463 \text{ ft.}$$

6.

This gives a time of about 445 s.

We have already found the times at the two endpts.

For $x = 0$, it was 525 s

& for $x = 2000$ it was 463 s.

Hence the min time is 445 s &
this happens if Alina walks 1463 feet along the sidewalk & then

$\sqrt{(2000-1463)^2 + 600^2} \approx 805$ ft

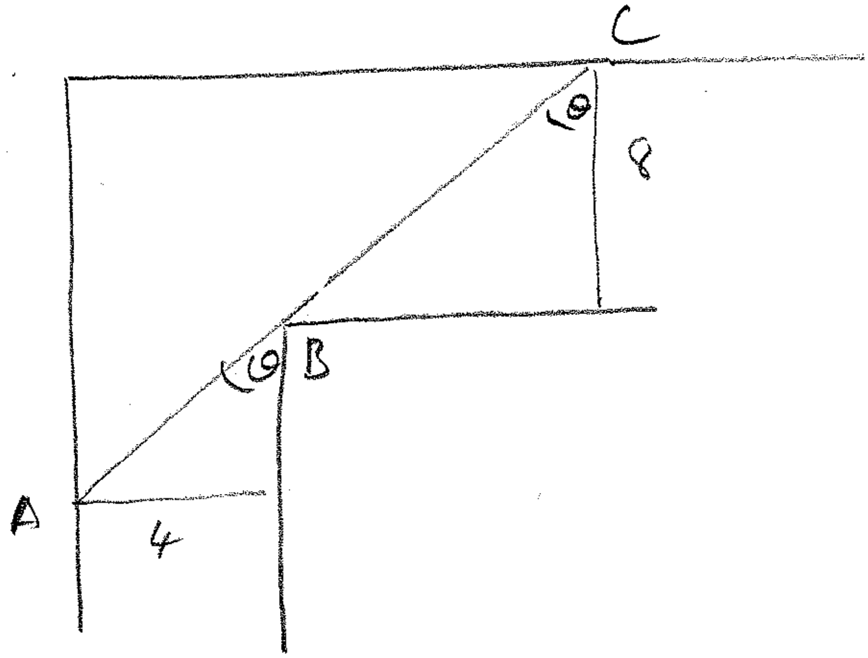
along the grass to the bus stop.

7. Seems reasonable!

Ex 3. One hallway which is 4ft wide meets another which is 8 ft wide. What is the length of the longest ladder which can be carried horiz. around the corner?

0.

1. To allow the longest ladder possible we carry the ladder around the corner so that it just touches both walls (at A & C) and the corner (at B).



The min such length would be the shortest ladder which would fit round the corner. ~~A~~ smaller ladder would still work (it would not touch A, B, and C simultaneously), but a larger one would not fit.

2. Let L be the length of the ladder.

We want to <u>minimize</u> L.

Have

$$L = \frac{4}{\sin\theta} + \frac{8}{\cos\theta}$$

$$= 4\operatorname{cosec}\theta + 8\sec\theta$$

3. No need to eliminate unwanted variables here.

4. Here $0 < \theta < \frac{\pi}{2}$.

5. Crit pts.

$$L = 4\csc\theta + 8\sec\theta$$

$$\frac{dL}{d\theta} = -4\csc\theta\cot\theta + 8\sec\theta\tan\theta = 0$$

Get

$$\frac{8\sin\theta}{\cos\theta \cdot \cos\theta} = \frac{4\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

$$\frac{\sin^3\theta}{\cos^3\theta} = \frac{1}{2}$$

$$\tan^3\theta = \frac{1}{\sqrt[3]{2}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt[3]{2}}\right) \approx 0.67 \text{ rad.}$$

Since $L \to \infty$ as $\theta \to 0$
and as $\theta \to \frac{\pi}{2}$,

this crit pt must be a min

& the min value of L is then

6. $4\csc(0.67) + 8\sec(0.67) \approx 16.65$ ft.

7. ✓