4.3 Optimization – Global Maxima and Minima.

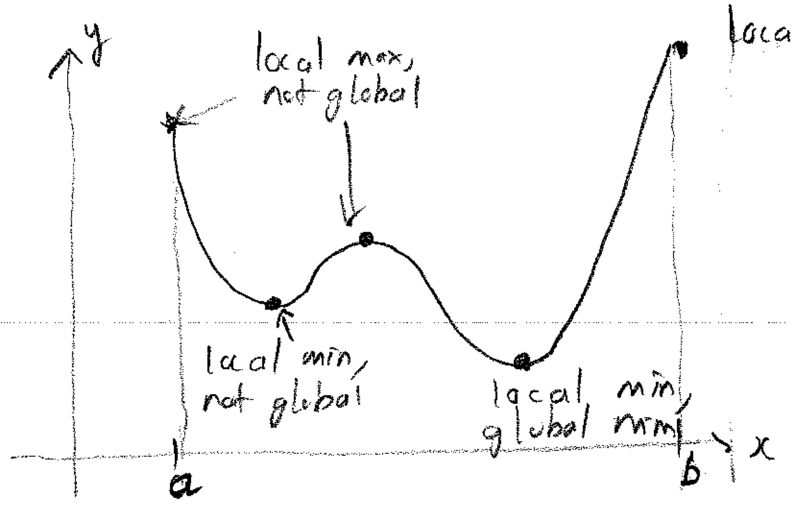
Global Maxima and Minima

- f has a global minimum at p if $f(p)$ is less than or equal to all other values of f.

- f has a global maximum at p if $f(p)$ is greater than or equal to all other values of f.

Often just called the minimum or maximum of f.

Theorem 4.2 If f is cts on a closed interval $[a,b]$, then it has a global min and a global max on $[a,b]$.

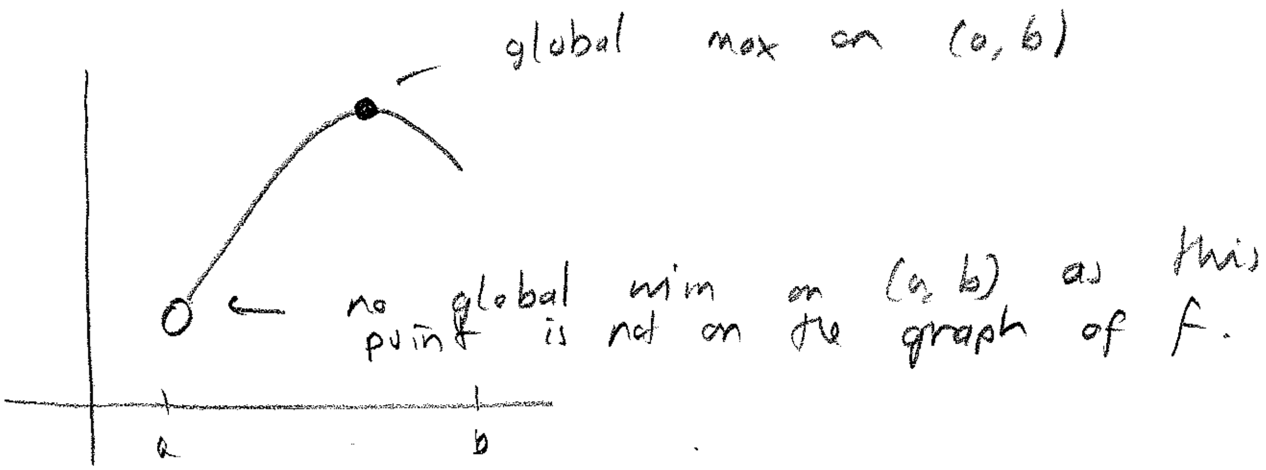


Moral Max or min can either be crit pts or end pts.

To find the min/max of f cts on a closed interval $[a,b]$

Compare values of the fn at all candidate pts: crit pts and endpts.

For open intervals or $\mathbb{R}$, situation is a little different



To find the min/max of f cts on an open interval or on $\mathbb{R}$.

Find the value of f at all of the crit pts & sketch a graph. Look at the values of f as x approaches the endpts. of the interval or $\pm\infty$ as appropriate.

Ex. 1 Find the global maxima and minima of $f(x) = x^3 - 9x^2 - 48x + 52$

on

a) $[-5, 12]$, b) $[-5, 0]$, c) $[-5, \infty)$.

Already found crit pts. at $-2, 8$;

$f(-2) = 104$, $f(8) = -396$.

a). Crit. pt.

$f(-2) = 104$, $f(8) = -396$.

End pt.

$f(-5) = -58$

$f(12) = -92$.

Comparing, we see that the global max on $[-5, 12]$ is 104 & occurs at $x = -2$ & the global min on $[-5, 12]$ is -396 & occurs at $x = +8$.

b) For $[-5,0]$, -2 is the only crit pt in this interval and

$$f(-2) = 104.$$

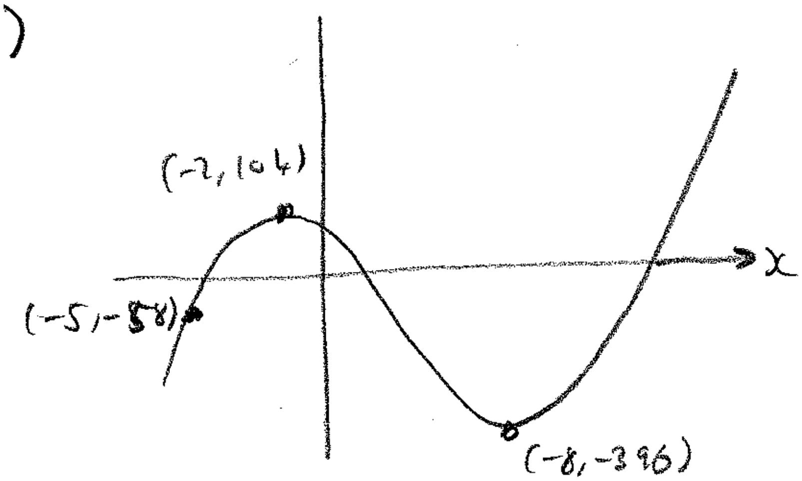
At the endpts

$$f(-5) = -58$$

$$f(0) = 52.$$

Comparing, the max value of f on $[-5,0]$ is 104 & occurs at $x=-2$. The min value of f is -58 & occurs at $x=-5$.

c)



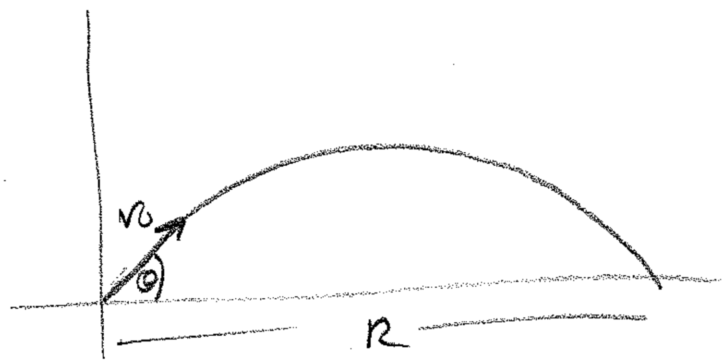
Picture shows that on $[-5,\infty)$ there is no maximum (can make $f(x)$ as large as we please by choosing x suff large and positive).

However the global min is -396 & occurs at $x=8$.

Ex2 When an arrow is shot into the air, its horizontal range R is given by

$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

where v_0 is the initial speed and θ_0 is the angle to the vertical at firing. Find the maximum value of R & for what angle it occurs.



Physically $0 \leq \theta \leq \frac{\pi}{2}$, so we maximize on this interval.

First find crit pts

$$R(\theta) = \frac{v_0^2 \sin(2\theta)}{g}$$

So $$R'(\theta) = \frac{2 v_0^2 \cos(2\theta)}{g}$$

$R'(\theta) = 0 \Rightarrow \cos(2\theta) = 0$

$\Rightarrow \theta = \ldots -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

Only soln in $[0, \frac{\pi}{2}]$ is $\underline{\frac{\pi}{4}}$.

$$R\left(\frac{\pi}{4}\right) = \frac{v_0^2 \sin\left(\frac{2\pi}{4}\right)}{g} = \frac{v_0^2 \cdot \sin\left(\frac{\pi}{2}\right)}{g} = \frac{v_0^2}{g}.$$

End pt.

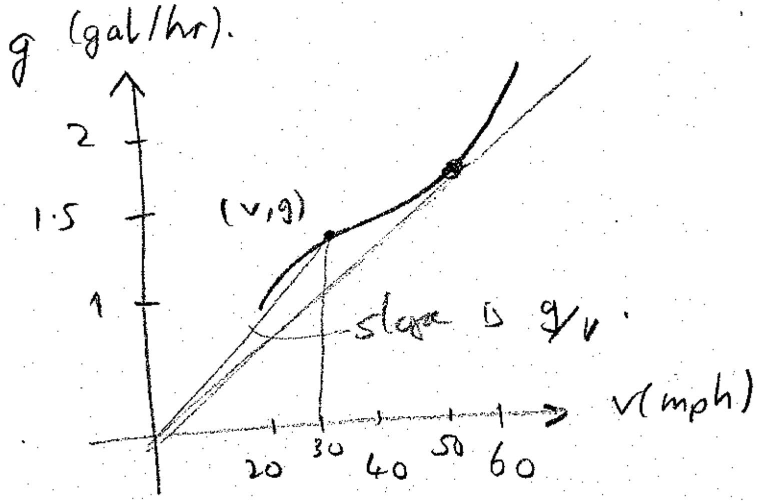
$R(0) = 0$

$R(\frac{\pi}{2}) = 0$

Hence the max. value of R is $\frac{v_0^2}{g}$ & this is attained for $\theta = \frac{\pi}{4} = 45°$.

Ex. Gas consumption.

The gas consumption g in gal/hr of a certain car is given as a fn of speed v in mph by the following graph.



Find the speed at which one consumes the least gas, ie for which we have the best mpg.

$\frac{g}{v}$ has units $\frac{\text{gal/hr}}{\text{miles/hr}}$ = gal/mile

Hence in order to get the best mpg, we want the minimum gal/mile

In other words, we want to minimise the slope g/v of the line connecting $(0,0)$ to (v,g).

From the graph, this happens at about 50mph.
where $g \sim 1.6$ gal/hr.

Here we get $\frac{50}{1.6} \approx 30$ mpg.

Upper and Lower Bounds

M is an upper bound for f is
$f(x) \leq M, \ \forall x$ in the domain of f.

m is a lower bound for f if
$f(x) \geq m \ \ \forall x$ in the domain of f.

e.g. For $f(x) = x^3 - 9x^2 - 48x + 52$ on $[-5, 12]$,
we saw $-396 \leq f(x) \leq 104$.
So -396 is a lower bound and
104 is an upper bound.

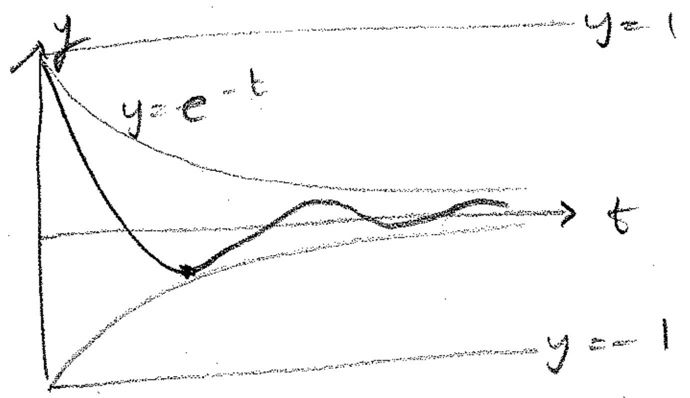
Game is to find the best possible bounds
– i.e. the least upper bound
and the greatest lower bound.

Ex 4. An object on a spring oscillates about its equil pos. at $y = 0$. Its distance from equil is given as a fn of time by

$$y = e^{-t} \cos t.$$

Find the greatest distance the object goes above & below the equil. for $t \geq 0$.

Soln. We are looking for bounds for y.



Graph is an exp decaying cosine curve with amplitude e^{-t}.

Since $0 \leq e^{t} \leq 1$ for $t \geq 0$

and $-1 \leq \cos t \leq 1$, we have

$$-1 \leq e^{-t} \cos t \leq 1, \quad t \geq 0.$$

The line $y = 1$ is the best possible upper bound as $y(0) = 1$.

To find a better (bigger) lower bound, we look for crit pts.

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^{-t}\cos t\right).$$

$$= \frac{d}{dt}(e^{-t}) \cdot \cos t + e^{-t}\frac{d}{dt}(\cos t).$$

$$= -e^{-t}\cos t + e^{-t} \cdot -\sin t$$

$$= -e^{-t}(\sin t + \cos t).$$

Since $e^{-t} > 0$, this means

$$\sin t + \cos t = 0$$

ie $\quad \sin t = -\cos t$

$$\frac{\sin t}{\cos t} = -1$$

$$\tan t = -1.$$

First place in $[0, \infty)$ this occurs
is at $t = \frac{3\pi}{4}$.

Here $y = e^{-3\pi/4} \cos\left(\frac{3\pi}{4}\right) \approx -0.067$.

and all other later soln's where $\tan t = 1$
will give larger values of y (why?).
Hence the global min for y is -0.067
and

$$-0.067 \leq e^{-t}\cos t \leq 1 \quad \forall t \geq 0.$$