Chapter 4 Using the Derivative

§4.1 Using First and Second Derivatives

Earlier we had the following.

If $f' > 0$ on an interval then f is incr on that interval.

If $f' < 0$ - - - - decr - - -

If $f' = 0$ - - - const.

If $f'' > 0$ on an interval, then f is concave up on that interval

If $f'' < 0$ on an interval, then f is concave down on that interval

If $f'' = 0$ on an interval, then f is linear on that interval.

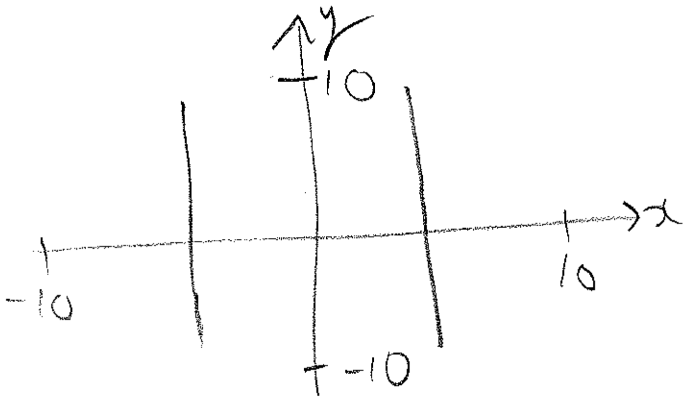
We can use the derivative to tell us more than this about a fn.

<u>Ex1</u> Sketch a nice graph of

$f(x) = x^3 - 9x^2 - 48x + 52.$

Since f is a cubic, expect an S-shaped graph.

If we just plot in the range $-10 \le x \le 10$, get something like.



Let's look at the derivative

$$f'(x) = 3x^2 - 18x - 48$$

To find where $f' > 0$ and $f' < 0$, first find where $f' = 0$.

Factoring, we get

$$f'(x) = 3(x+2)(x-8)$$

The sign of f' is controlled by the signs of the two factors $(x+2)$ and $(x-8)$.

$(x+2)$ changes sign at -2

and $(x-8)$ changes sign at 8.

Make a table.

Interval	$(x+2)$	$(x-8)$	f'	incr/decr
$(-\infty, -2)$	< 0	< 0	> 0	incr.
$(-2, 8)$	> 0	< 0	< 0	decr.
$(8, \infty)$	> 0	> 0	> 0	incr.

f incr	$x=-2$	f decr	$x=8$	f incr
$f'>0$		$f'<0$		$f'>0$

Find $f(-2) = 104$, $f(8) = -396$

Hence on $(-2, 8)$, f decr from 104 to -396.

Also easy to find the y-intercept

$f(0) = 52$.

Now look at concavity. For this we need f''.

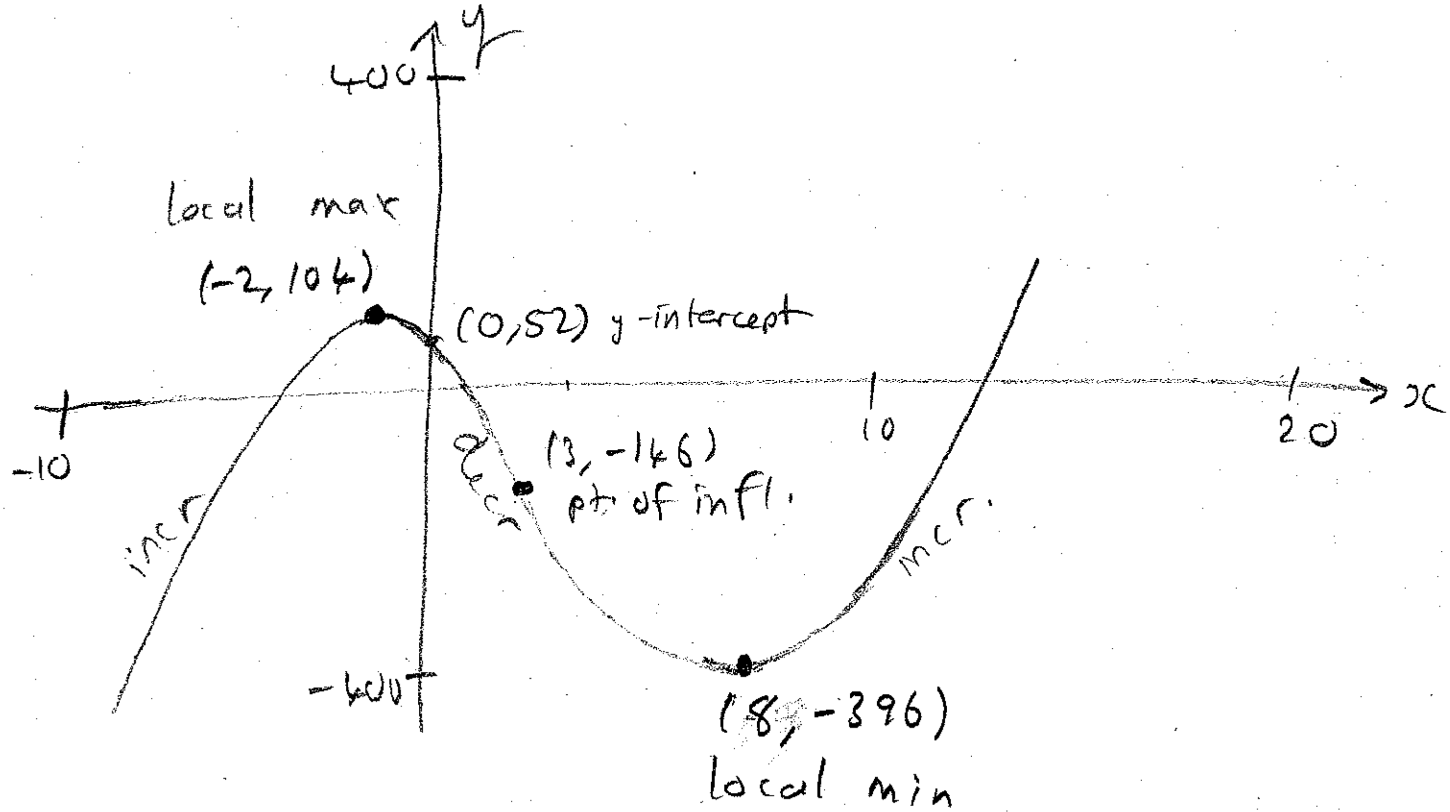
$$f''(x) = 6x - 18$$
$$= 6(x-3) \quad \text{- changes sign at } 3.$$

Again, make a table.

interval	$(x-3)$	$f''(x)$	concavity
$(-\infty, 3)$	< 0	< 0	down
$(3, \infty)$	> 0	> 0	up.

Here $x = 3$ is a pt of infl.

Can now make a sketch of the graph.



Here -2 was called a local max, and 8 a local min for f.

In general, for any fn f and a pt p in the domain of f.

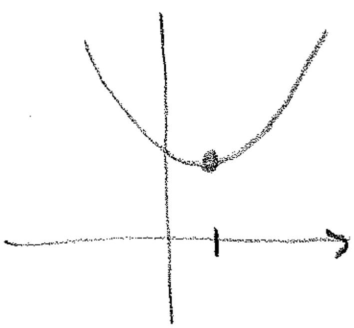
- f has a <u>local minimum</u> at p if f(p) is less than or equal to the values of f for points near p
- f has a <u>local maximum</u> at p if f(p) is greater than or equal to the values of f for points near p.

How do we Detect Local Max/Min?

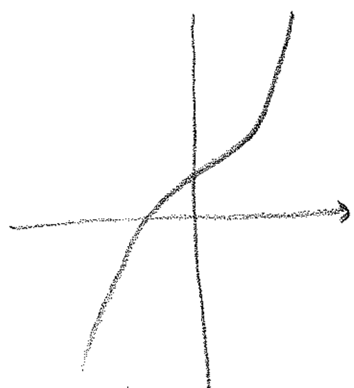
<u>Defn</u>

For any fn f, a pt p in the domain of f where $f'(p) = 0$ or <u>$f'(p)$ is undefined</u> is called a <u>critical point</u> of f. The pt $(p, f(p))$ on the graph of f is also called a critical point while the number $f(p)$ is called a <u>critical</u> <u>value</u>.

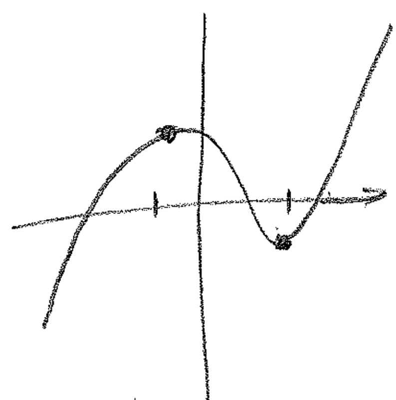
A fn may have any number of crit pts or none at all.



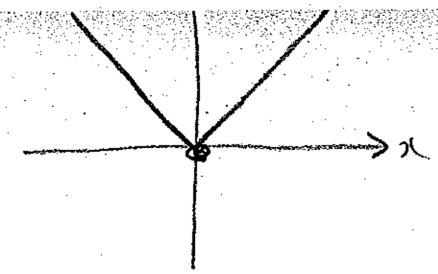
quadratic
- one crit pt.



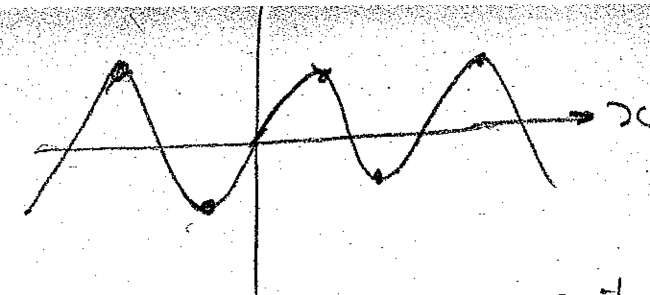
cubic
no crit pt.



cubic
2 crit pts



$|x|$ - one crit pt.



$\sin x$ many crit pts.

What is the point of critical points?

Crit pts divide the domain of f into intervals on which the sign of the derivative f' remains the same. On each of these intervals, the graph of f cannot change direction and f is either incr or decr.

<u>Theorem 4.1</u> (Fermat's Theorem) Local Extrema and Critical Points.

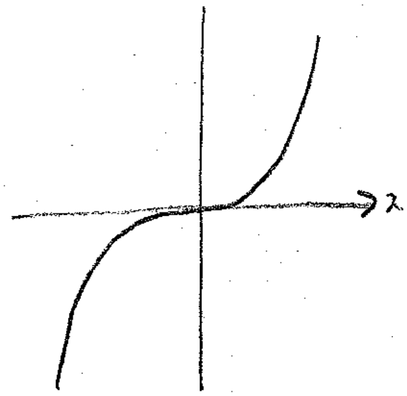
Suppose f is defined on an interval and has a local max or min at $x=a$ which is not an endpt of the interval. Then either f is not diff at a or $f'(a)=0$. In either case a is a crit pt.

Roughly speaking, Fermat says

local max/min $\Rightarrow$ crit pt.

The converse is not true. $\boxed{\text{TRAP!}}$

e.g. $f(x) = x^3$



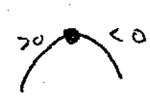
$f'(0) = 0$, but 0 is neither a local min nor a local max.

Testing for Local Maxima and Minima

First Method - The First Derivative Test

Suppose p is a crit pt of a cts fn f.

- If f' changes from negative to positive at p, then f has a local min at p. 

- If f' changes from positive to negative at p, then f has a local max at p. 

Ex2. Use the first deriv test to confirm $f(x) = x^3 - 9x^2 - 48x + 52$ has a local max at -2 & a local min at 8.

Recall that $f'(x) = 3x^2 - 18x - 48 = 3(x+2)(x-8)$.

When we did our table we found that f' went from +ve to -ve at $x = -2$. Hence this is a local max by the first deriv test.

f' went from -ve to +ve at $x = 8$, so this was a local min by the first deriv test.

Ex 3. $f(x) = \dfrac{1}{x(x-1)}$

Diff. $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{x(x-1)}\right)$

$$= \frac{d}{dx}\left(\frac{1}{x^2-x}\right)$$

$$= \frac{d}{dx}(x^2-x)^{-1}$$

chain rule $= (-1)(x^2-x)^{-2}\dfrac{d}{dx}(x^2-x)$

$$= -(x^2-x)^{-2} \cdot (2x-1)$$

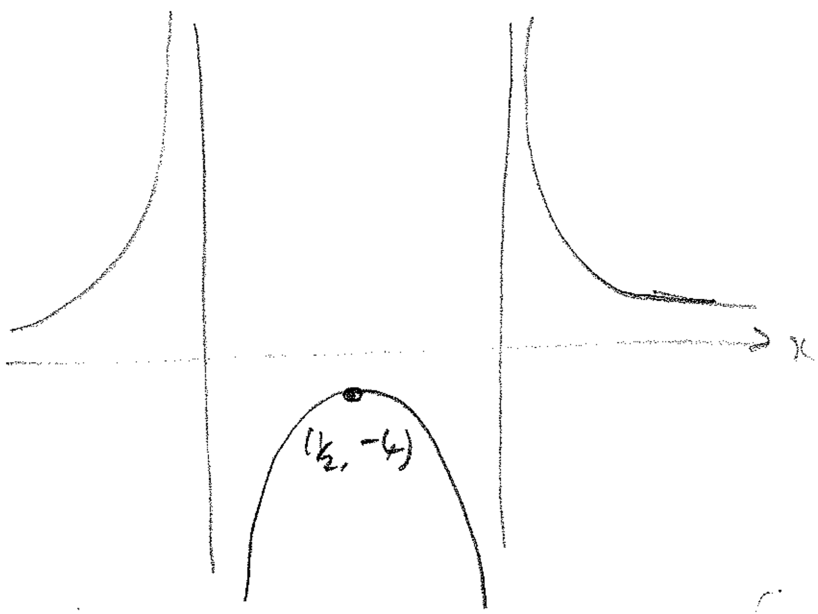
$$= -\frac{2x-1}{(x^2-x)^2}.$$

Bottom factor is squared and so always ≥ 0.
Hence the sign of f' is controlled by
$2x-1$ which vanishes at $x = \frac{1}{2}$.

Interval	$2x-1$	f'	Incr/Decr.
$(0, \frac{1}{2})$	< 0	> 0	Incr
$(\frac{1}{2}, 1)$	> 0	< 0	Decr.

Hence by the first deriv. test, f has a local max at $x = \frac{1}{2}$.

Graph looks like



Note that although f is decr. everywhere it is defined on $(\frac{1}{2}, \infty)$, f is not incr on this whole interval. Problem is that there is a vertical asymptote at $x = 1$ at which f is not defined.

Ex 3. Explain why $\sin x + 2e^x$ has no local max or min for $x > 0$.

$$\frac{d}{dx}(\sin x + 2e^x) = \cos x + 2e^x.$$

For $x \geq 0$, $-1 \leq \cos x \leq 1$

while $2e^x \geq 2e^0 = 2$.

Hence $\cos x + 2e^x \geq -1 + 2 = 1 > 0$.

Hence $f' > 0$ on $(0, \infty)$ and so f is incr and there is no local max or min here.

Second Method

The Second Derivative Test

- If $f'(p)=0$ and $f''(p)>0$, then f has a local min at p.
- If $f'(p)=0$ and $f''(p)<0$, then f has a local max at p.
- If $f'(p)=0$ and $f''(p)=0$, then the test tells us nothing.

Ex 4. $f(x) = x^3 - 9x^2 - 48x + 52$ again.

Had $f'(x) = 3x^2 - 18x - 48$

$f''(x) = 6x - 18.$

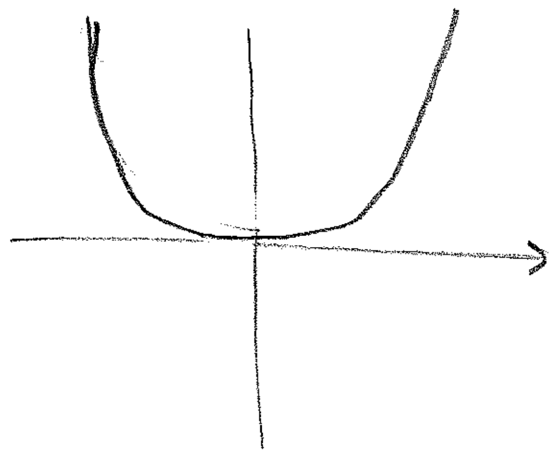
$f'(x) = 3(x+2)(x-8)$

so $f'(x)=0$ at $x=-2$, $x=+8$.

$f''(-2) = -30 < 0$ so -2 is a local max for f by 2nd deriv test

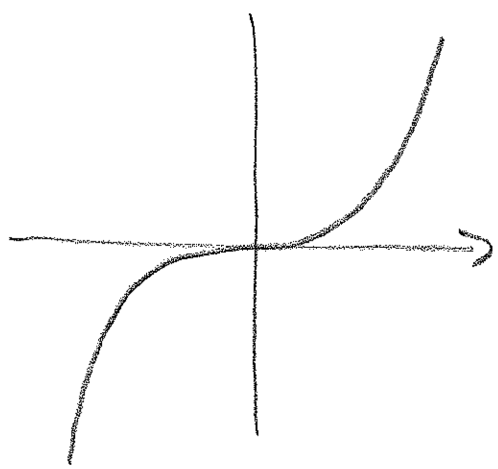
$f''(8) = 30 > 0$ so 8 is a local min for f by the second deriv. test

Ex 5. Why the test tells us nothing when $f''(p) = 0$.



$g(x) = x^4$

$g''(0) = 0$ - local min.



$h(x) = x^3$

$h''(0) = 0$ neither local min nor local max.

Concavity and Points of Inflection

Recall that a pt. of inflection was a point where the concavity changed (up to down or down to up).

How to Spot Them

Since concavity is (usually) governed by the sign of f'', a pt. where f'' changes sign is a pt. of inflection.

This could be either a place where $f''=0$ or f'' is undefined.

Hence to look for pts. of inflection, we look for places where either $f''=0$ or f'' does not exist.

Ex 6. Find the crit & infl. pts. for $g(x) = x e^{-x}$, and sketch its graph for $x \geq 0$.

Taking derivatives and simplifying,

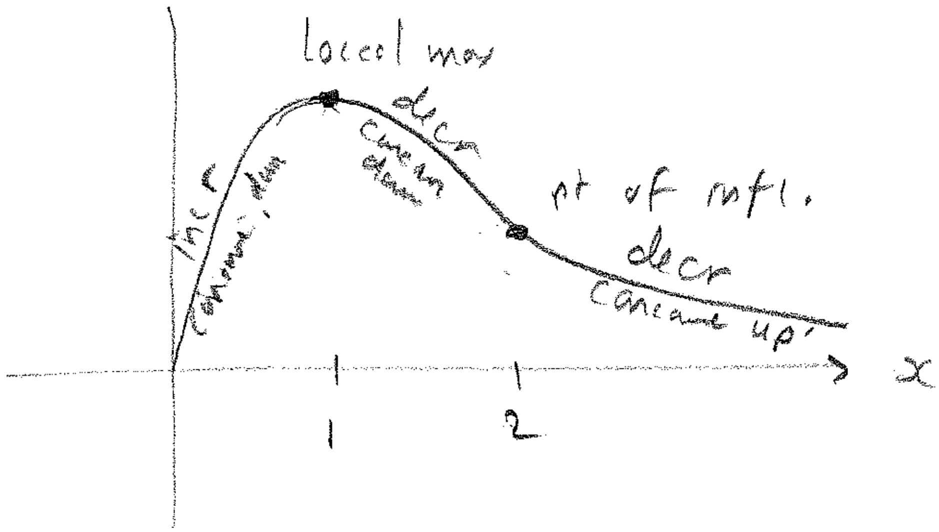
$$g'(x) = -(x-1)e^{-x}$$

$$g''(x) = (x-2)e^{-x}.$$

g has a crit pt at $x=1$ and $g''(+1) = (1-2)e^{-1} = -e^{-1} < 0$, so this is a local max by the second deriv test.

$g''(2) = 0$ and $g''(x) < 0$ for $x < 2$
> 0 for $x > 2$.

Hence the concavity of g changes at $x = 2$ and this is a pt of infl.

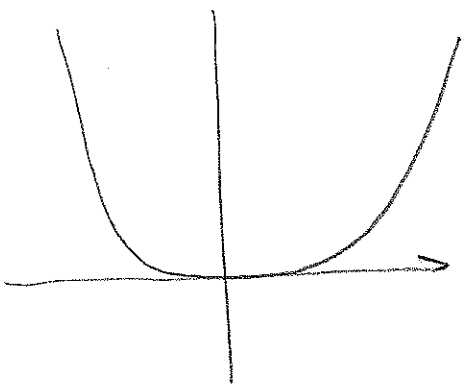

local max
decr
concave down
pt of infl.
decr
concave up
incr
concave down
x
1
2

WARNING (AGAIN!)

Have

pt of inf $\Rightarrow$ $f'' = 0$ or f'' undefined.

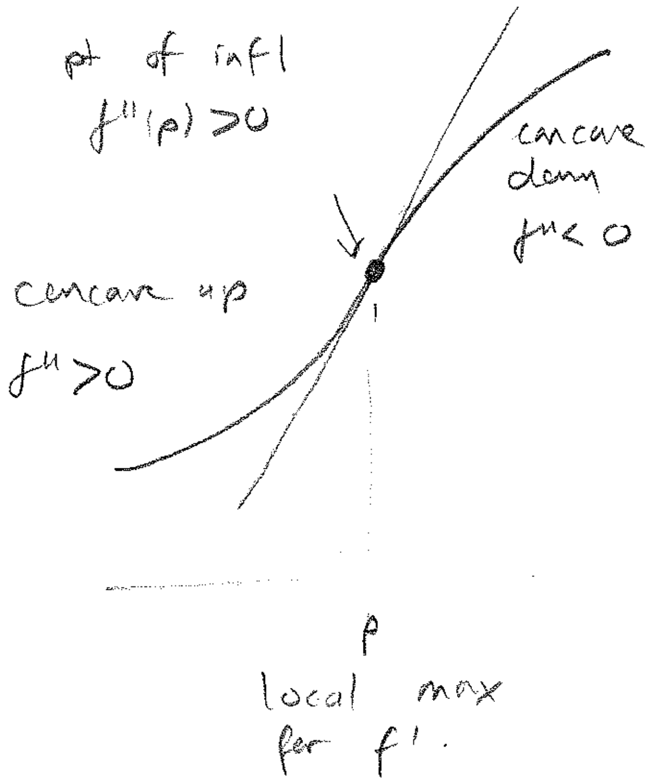
Converse is not true

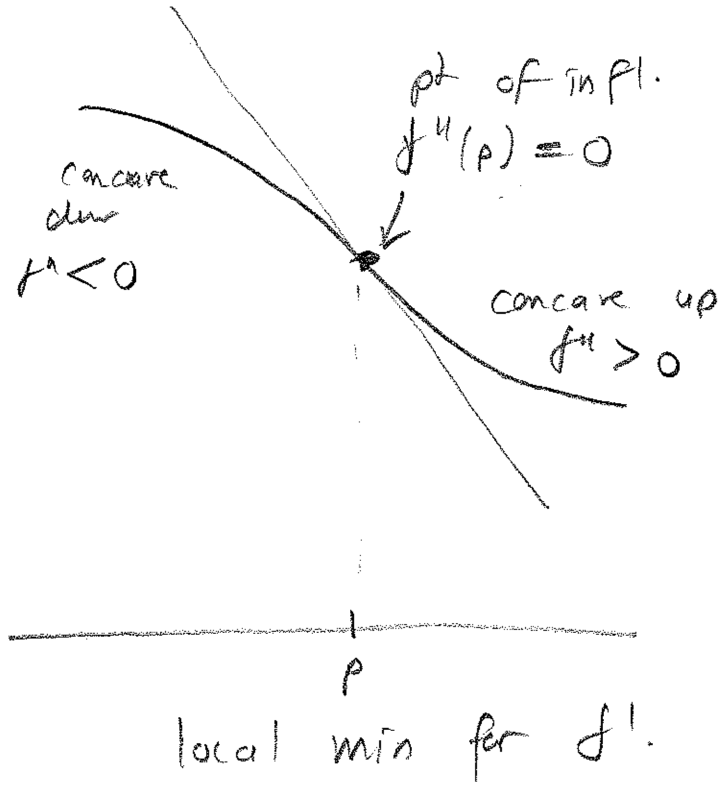


$f(x) = x^4$ $\quad$ $f''(0) = 0$ but 0 is a local min, not a pt of inflection.

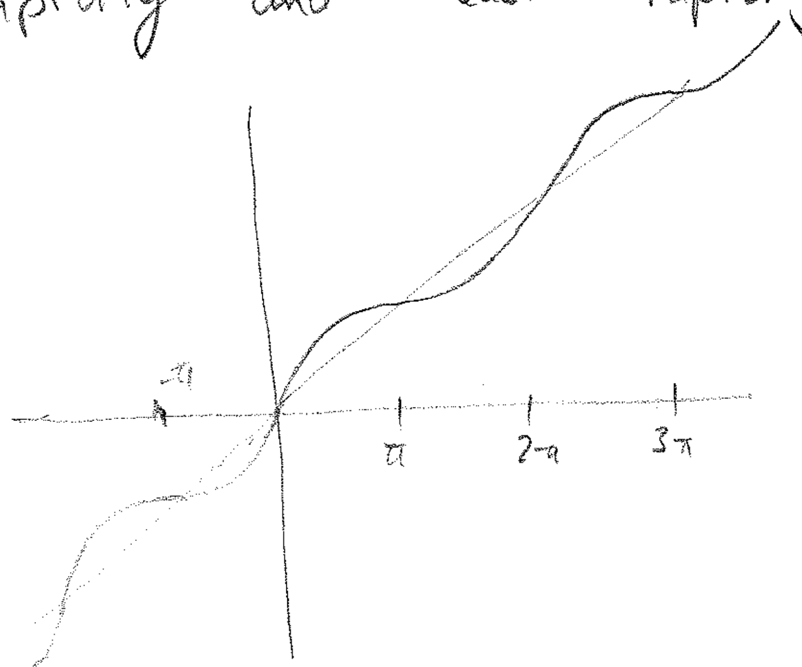
Inflection Points and Local Maxima and Minima of the Derivative

Suppose a fn f has a cts. derivative f' (i.e. f is C^1). If f' changes sign at p, then f has a pt. of infl at p and the derivative f' has a local max or local min at p.





Ex Graph $f(x) = x + \sin x$
and determine where f is incr most rapidly and least rapidly.



$f(x) = x + \sin x$

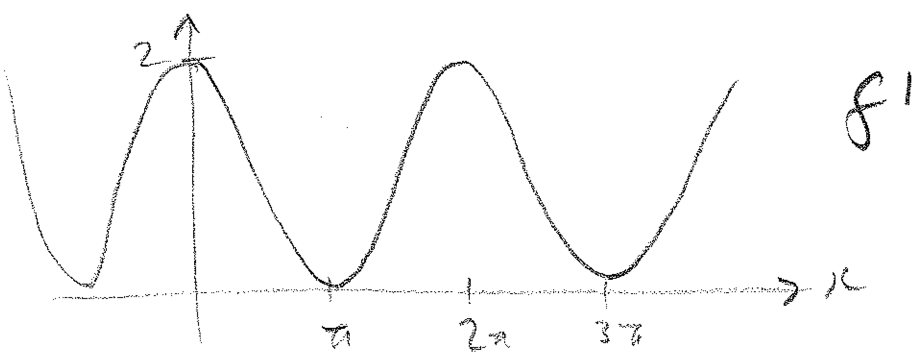
$f'(x) = 1 + \cos x$

f is incr most rapidly
when $\cos x = 1$, i.e at $x = \ldots -2\pi, 0, 2\pi, 4\pi \ldots$
Here $f' = 2$

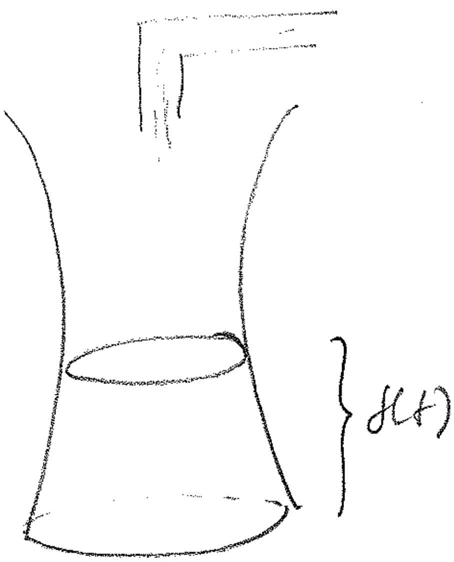
f is incr least rapidly
when $\cos x = -1$ ie at $x = \ldots -\pi, \pi, 3\pi, \ldots$

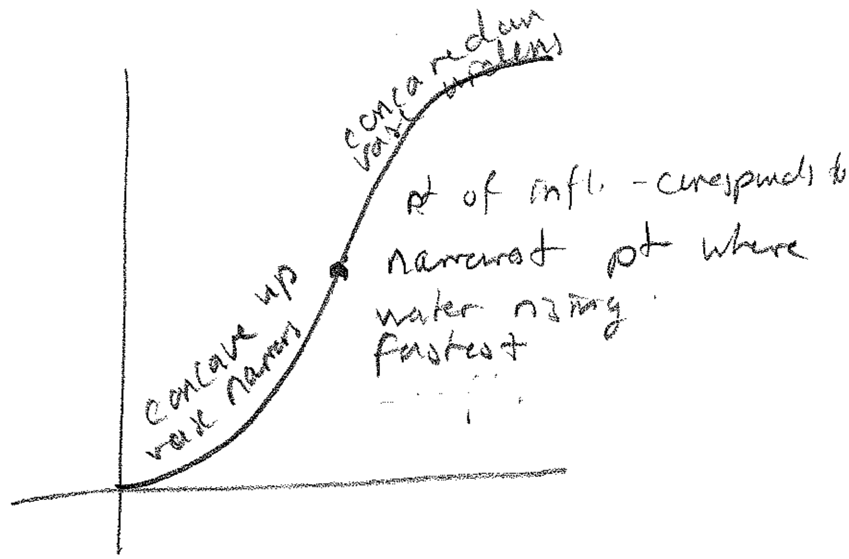
Here $f' = 0$.

Both sets of points are points of infl.



Ex. Water is being poured into the vase as shown at a constant rate. Make a graph of $f(t)$, the depth of water as a fn of time and explain the concavity & pts of infl.





At the start the base is wide, so the water level rises slowly. As the vase narrows the level rises faster and is rising fastest. Thereafter the vase gets wider and the water rises more slowly again.

The narrowing part corresponds to $f(t)$ being concave up and the widening part to $f(t)$ being concave down. At the pt of infl. $f'(t)$ is at its max and this corresponds to the narrowest part of the vase.