§ 3.10 Theorems about Differentiable Functions

Theorem 3.7 Mean Value Theorem.

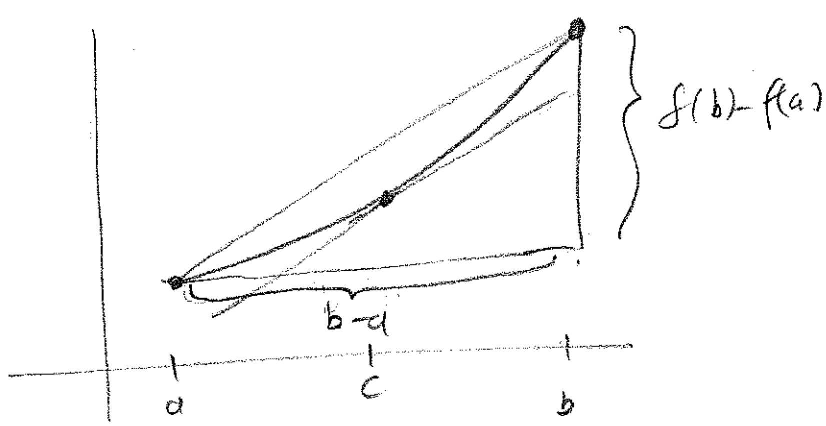
If f is cts on $[a, b]$ and diff on (a, b), then $\exists\, c \in (a, b)$ $(a < c < b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently
$f(b) - f(a) = (b-a) f'(c)$.

$$f(b) = f(a) + (b-a) f'(c).$$

Interpretation.



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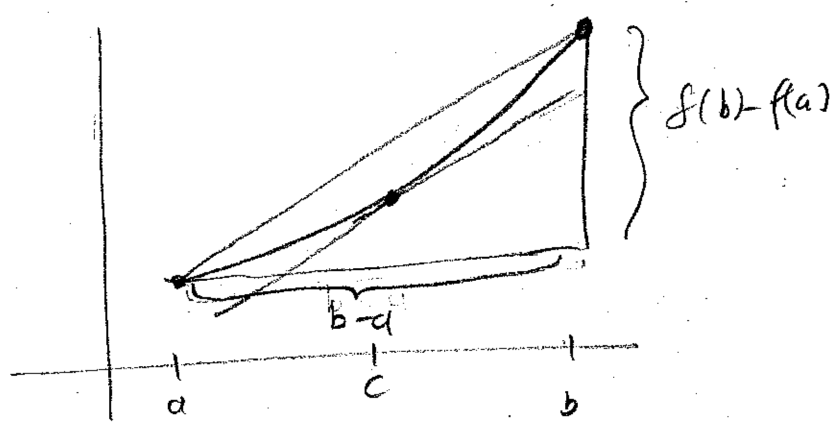


$\dfrac{f(b)-f(a)}{b-a}$ is the slope of the secant line passing through the pts $(a, f(a))$, $(b, f(b))$.
- a sort of crude derivative.

What MVT says is that at some point strictly between a & b, the slope of the tgt line at this pt is the same as the slope of this secant line.

Can use MVT to prove the following.

<u>Theorem 3-8</u> Incr. Fn Thm.

Suppose f cts on $[a,b]$, diff on (a,b).

- If $f'(x) > 0$ on (a,b), then f is incr on $[a,b]$
- If $f'(x) \geq 0$ on (a,b), then f is nondecr on $[a,b]$.

Theorem 3.9 The Constant Function Theorem

Supp f cts on $[a,b]$, diff on (a,b).
If $f'(x) = 0$ on (a,b), then f is const on $[a,b]$.

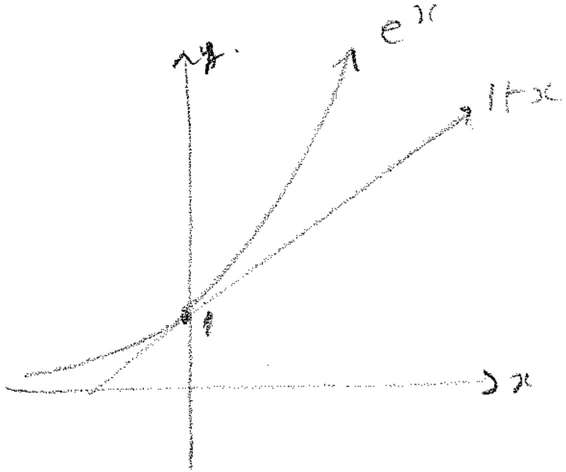
Theorem 3.10 The Racetrack Principle

Supp g, h cts on $[a,b]$, diff on (a,b) and that $g'(x) \leq h'(x)$ for $a < x < b$.

- If $g(a) \leq h(a)$, then $g(x) \leq h(x)$ on $[a,b]$.
- If $g(b) \geq h(b)$, then $g(x) \geq h(x)$ on $[a,b]$.

Interpretation in terms of race horses.

Ex 1. Explain graphically why $e^x \geq 1 + x$ $\forall x$ and then prove it by racetrack.



e^x is concave up on $\mathbb{R}$ & the tgt line at $x=0$ is $y=1+x$. Hence graphically $e^x \geq 1+x$ on $\mathbb{R}$.

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(1+x) = 1$$

$e^x \geq 1$ on $(0,\infty)$

& $e^0 = 1$ so

$e^x \geq 1+x$ for $x \geq 0$.

$e^x \leq 1$ on $(-\infty,0)$

& $e^0 = 1$ so

$e^x \geq 1+x$ for $x \leq 0$.