§ 3.9 Linear Approximations and the Derivative

Recall If ~~a fn~~ f is diff at a, then the graph of f near a looks like a straight line. The eq^n of the tgt line to the graph of f at (a, f(a)) is

$$y = f(a) + f'(a)(x-a).$$

If f is diff at $x=a$, then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a),$$

or

$$\lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a).$$

So if x is close to a,

we have

$$\frac{f(x) - f(a)}{x-a} \approx f'(a)$$

$$\Rightarrow \quad f(x) - f(a) \approx f'(a)(x-a)$$

ie $$f(x) \approx f(a) + f'(a)(x-a)$$

This is called the <u>tangent line approximation</u> for f near $x = a$ and the expression on the right is called the <u>local linearization</u> of f near $x=a$.

Now let's consider the difference

$$E(x) = f(x) - f(a) - f'(a)(x-a).$$

This is the error in the tgt line approx.

Let's look at $\frac{E(x)}{x-a}$ for x near a but $x \neq a$

$$\frac{E(x)}{x-a} = \frac{f(x) - f(a) - f'(a)(x-a)}{x-a}$$

$$= \frac{f(x) - f(a)}{x-a} - f'(a).$$

As $x \to a$, this tends to

$$f'(a) - f'(a) = 0.$$

Hence we have shown.

Thm 3.6 Differentiability and Local Linearity

The error, $E(x)$, in the tgt. line approx satisfies.

$$\lim_{x \to a} \frac{E(x)}{x-a} = 0.$$

Clearly this must mean that

$$\lim_{x \to a} E(x) = 0$$

what theorem 3.6 really tells us is that as $x \to a$, $E(x)$ tends to 0 faster than $(x-a)$ does. (e.g. like $(x-a)^2$).

Ex 1. What is the tgt line approx to $\sin x$ at $x = 0$?

Tgt line approx.

$$f(x) \approx f(0) + f'(0)(x - 0)$$

If $f(x) = \sin x$, $f'(x) = \cos x$
$f(0) = 0$, $f'(0) = 1$, so

$$\sin x \approx x.$$

Ex 2. What is the local linearization for e^{kx} near $x = 0$

$$f(x) = e^{kx}, \quad f'(x) = ke^{kx}$$

$$f(x) \approx f(0) + f'(0)(x-0)$$

Here $f(0) = 1$, $f'(0) = 1$, so the local linearization for e^{kx} is

$$1 + kx.$$

More on errors.

Suppose we try to find the tgt. line approx to $f(x)=x^2$ at $x=0$.

Here $\quad f(x) = x^2, \quad f'(x) = 2x$

$\quad f(0) = 0, \quad f'(0) = 0$

The tgt line approx is

$$f(x) \approx f(0) + f'(0)(x-0)$$

which in this case means

$$x^2 \approx 0 + 0(x-0)$$

ie

$$x^2 \approx 0.$$

Here clearly $\quad E(x) = x^2.$

In general, (provided $f''(a)$ exists),

$$E(x) \approx \frac{f''(a)}{2}(x-a)^2.$$

For $\sin x$, $a=0$, $f''(0) = 2$

and

$$E(x) = \frac{f''(0)}{2}(x-0)^2.$$

Now use Maple