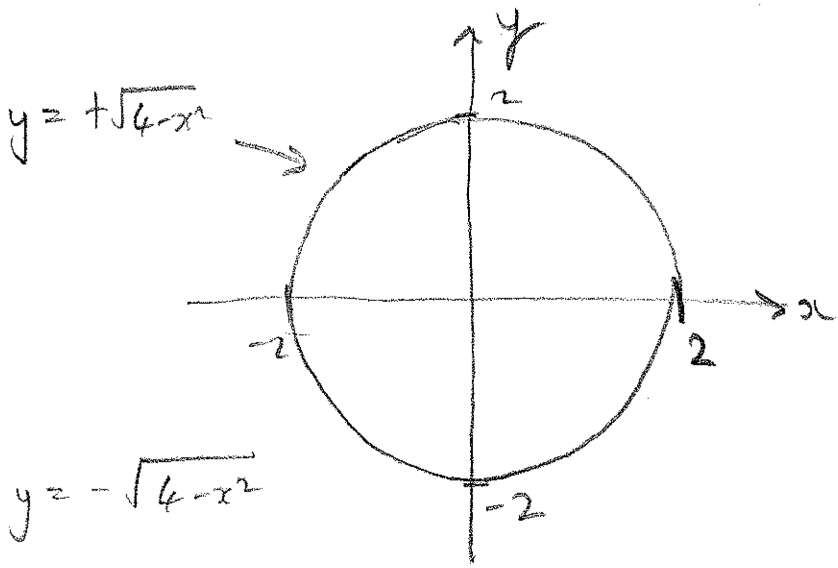
§ 3.7 Implicit Functions

Main idea : We can find $\frac{dy}{dx}$ even when y isn't a function of x.

Ex $\quad x^2 + y^2 = 4$

Circle of radius 2 about $(0,0)$.



$$y^2 = 4 - x^2$$

So $\quad y = \pm\sqrt{4-x^2}$

y is not a $f^{\underline{n}}$ of x here because of the two possible signs before the $\sqrt{\ }$.

Let's return to our original eq^n.

$$x^2 + y^2 = 4.$$

If we treat y <u>locally</u> as a fn of x and use the chain rule we get:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} = 0$$

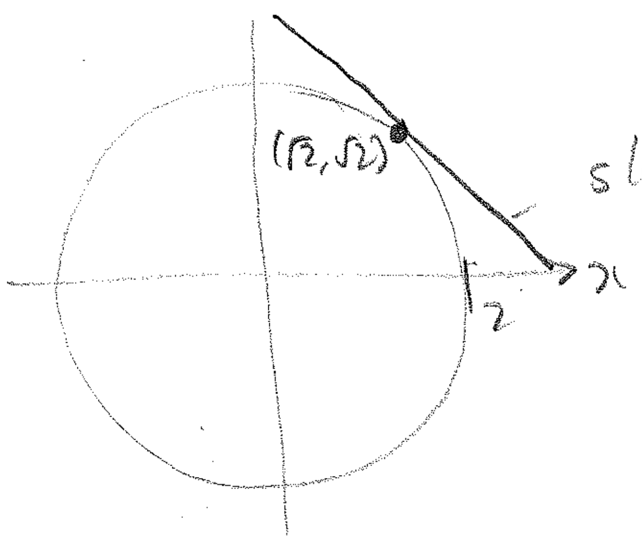
$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad (\text{provided } y \neq 0).$$

Drawback The answer is in terms of x and y - not just x.

However, if we know a particular point (x,y) on the curve, then we can calculate the slope of the tangent line at that point.

e.g. $(\sqrt{2},\sqrt{2})$ is on $x^2+y^2=4$ and the slope at this point is $-\frac{x}{y} = -\frac{\sqrt{2}}{\sqrt{2}} = -1$



This kind of differentiation is called implicit differentiation.

Ex 1. $y^3 - xy = -6$

Again, regard y as locally a fn of x and diff impl. using the chain rule, and product rule.

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(xy) = \frac{d}{dx}(-6)$$

$$3y^2\frac{dy}{dx} - \left(1 \cdot y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}(3y^2 - x) = y$$

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$\frac{dy}{dx} = 0 \Rightarrow y = 0$ However, there is no point on the curve for which $y = 0$ as this would mean

$$y^3 - xy = -6$$
$$0 - 0 = -6 \ !$$

We have a vertical tangent when the denominator $3y^2 - x = 0$

ie when $x = 3y^2$.

Subst into

$$y^3 - xy = -6$$

$$y^3 - 3y^2 \cdot y = -6$$

$$-2y^3 = -6$$

$$y^3 = 3$$

$$y = \sqrt[3]{3} = 3^{\frac{1}{3}}$$

Then $x = 3y^2 = 3 \cdot (3^{\frac{1}{3}})^2 = 3 \cdot 3^{2/3} = 3^{5/3}$

and we have a vertical tangent

at $(3^{5/3}, 3^{\frac{1}{3}}) \sim (6.24, 1.44)$.

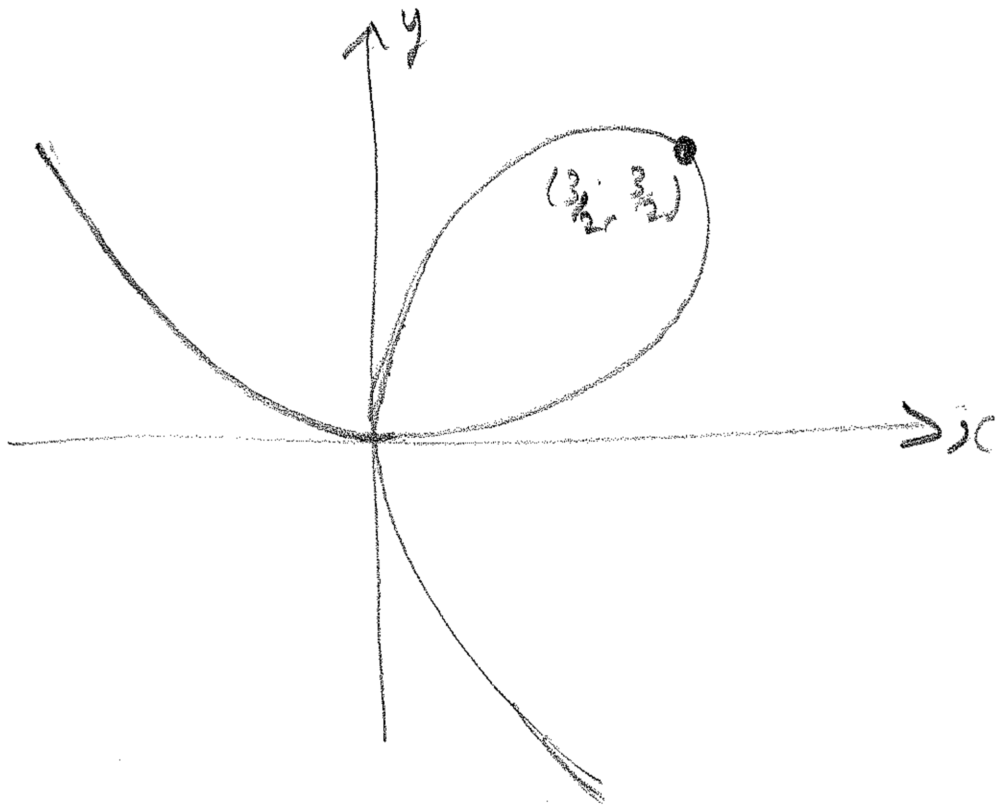
Ex 2. Find the eqn of the tgt line to the folium of Descartes

$$x^3 + y^3 = 3xy$$

at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

First check $\left(\frac{3}{2}, \frac{3}{2}\right)$ is on the curve.

Picture looks something like this



Use impl. diff

$$x^3 + y^3 = 3xy$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3y + 3x \frac{dy}{dx}$$

(chain rule) (prod. + chain rules).

Gather $\frac{dy}{dx}$ terms.

$$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$= \frac{y - x^2}{y^2 - x}$$

At $(\frac{3}{2}, \frac{3}{2})$

$$\frac{dy}{dx} = \frac{\frac{3}{2} - \frac{9}{4}}{\frac{9}{4} - \frac{3}{2}} = \frac{-\frac{3}{4}}{\frac{3}{4}} = \underline{-1}.$$

The eqn of the tgt line is then

$$y - y_0 = m(x - x_0)$$

$$y - 3/2 = -1(x - 3/2)$$

$$y = -x + 3$$