§ 3.6 The Chain Rule and Inverse Fns.

Finding the derivative of an inverse fn.: Derivative of $x^{\frac{1}{2}}$.

Consider $f(x) = x^{\frac{1}{2}} = \sqrt{x}$. $(x > 0)$.

Clearly $(f(x))^2 = x$ and so, differentiating both sides and using the chain rule

$$\frac{d}{dx}\left(f(x)\right)^2 = \frac{d}{dx}(x)$$

$$2f(x) \cdot f'(x) = 1.$$

So
$$f'(x) = \frac{1}{2f(x)}$$

$$= \frac{1}{2x^{\frac{1}{2}}}$$

$$= \frac{1}{2\sqrt{x}}$$ as we know from earlier.

Derivative of $\ln x$.

$\ln x$ is the inverse of e^x and so

$$e^{\ln x} = x \qquad (x>0).$$

Again, diff both sides and then use the chain rule.

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x)$$

$$e^{\ln x}\frac{d}{dx}(\ln x) = 1$$

ie. $x\,\frac{d}{dx}(\ln x) = 1$ as $e^{\ln x} = x$

So $$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x>0}$$

Ex1.

a) $\frac{d}{dx}(\ln(x^2+1))$

$\underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{1}{x^2+1} \frac{d}{dx}(x^2+1)$

$= \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$

b) $\frac{d}{dt}(t^2 \ln t)$

$\underset{\text{rule}}{\overset{\text{product}}{=}} \frac{d}{dt}(t^2) \cdot \ln t + t^2 \frac{d}{dt}(\ln t)$

$= 2t \cdot \ln t + t^2 \cdot \frac{1}{t}$

$= 2t \ln t + t.$

c) $\frac{d}{dy}\left(\sqrt{1+\ln(1-y)}\right)$

$= \frac{d}{dy}\left((1+\ln(1-y))^{\frac{1}{2}}\right)$

$\underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{1}{2}(1+\ln(1-y))^{-\frac{1}{2}} \frac{d}{dy}(1+\ln(1-y))$

$\underset{\text{rule}}{\overset{\text{chain}}{=}} \frac{1}{2}(1+\ln(1-y))^{-\frac{1}{2}} \cdot \frac{1}{1-y}\frac{d}{dy}(-y)$

$= \frac{1}{2}(1+\ln(1-y))^{-\frac{1}{2}} \cdot \frac{1}{1-y} \cdot -1$

$= \dfrac{-1}{2(1-y)\sqrt{1+\ln(1-y)}}$

Derivative of a^x. (again)

Already saw $\frac{d}{dx}(a^x) = \ln a \cdot a^x$.

Do this again using the identity

$$\ln(a^x) = x \ln a$$

Diff both sides wrt x & use the chain rule

$$\frac{d}{dx}(\ln(a^x)) = \frac{d}{dx}(x \ln a)$$

$$\frac{1}{a^x} \cdot \frac{d}{dx}(a^x) = \ln a$$

$$\frac{d}{dx}(a^x) = \ln a \bullet a^x.$$

Derivatives of Inverse Trigonometric Fns.

In §1.5, we defined arc sin x as the angle between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$ (incl.) whose sine is x.

Similarly, arc tan$x$ is the angle strictly between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$ whose tangent is x.

To find $\frac{d}{dx}(\text{arc tan}x)$ we use the identity

$$\tan(\text{arc tan}\,x) = x \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right).$$

As usual, we diff both sides wrt x & use the chain rule.

$$\text{So } \frac{d}{dx}(\tan(\text{arc tan}\,x)) = \frac{d}{dx}(x).$$

$$\sec^2(\arctan x) \cdot \frac{d}{dx}(\arctan x) = 1$$

Now remember that $\sec^2\theta = 1 + \tan^2\theta$
so that

$$(1 + \tan^2(\arctan x)) \cdot \frac{d}{dx}(\arctan x) = 1.$$

ie

$$(1 + x^2)\frac{d}{dx}(\arctan x) = 1$$

since $\tan(\arctan x) = x$.

So
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

Similarly
$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1.$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

$$\frac{d}{dx}(\text{arc} \cot x) = \frac{-1}{1+x^2}. \quad , x \in \mathbb{R}.$$

Ex 2.

a) $$\frac{d}{dt}(\arctan t^2) \overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{1}{1+(t^2)^2} \frac{d}{dt}(t^2).$$

$$= \frac{2t}{1+t^4}$$

b) $$\frac{d}{d\theta}(\arcsin(\tan\theta))$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} \frac{1}{\sqrt{1-\tan^2\theta}} \cdot \frac{d}{d\theta}(\tan\theta)$$

$$= \frac{\sec^2\theta}{\sqrt{1-\tan^2\theta}} = \frac{1}{\cos^2\theta\sqrt{1-\tan^2\theta}}$$

Derivative of a General Inverse Fn

Each of the prev. results gave the derivative of an inverse fn.

In general, if a diff fn f has a diff inverse f^{-1}, then we can find its deriv by differentiating

$$f(f^{-1}(x)) = x$$

& using the chain rule.

$$\frac{d}{dx}\left(f(f^{-1}(x))\right) = \frac{d}{dx}(x).$$

$$f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) = 1.$$

$$\boxed{\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}}$$

Ex. For $f(x) = 1 + x + x^3$, find $(f^{-1})'(3)$.

First $f(1) = 1 + 1 + 1^3 = 3$

So $f^{-1}(3) = 1$

$$f'(x) = \frac{d}{dx}(1 + x + x^3)$$

$$= 1 + 3x^2$$

So
$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$$

$$= \frac{1}{f'(1)}$$

$$= \frac{1}{1 + 3 \cdot 1^2}$$

$$= \frac{1}{4}.$$