§ 3.5 Derivatives of Trigonometric Functions

First some review.

Identities

$$\sin^2 x + \cos^2 x = 1, \qquad \sec^2 x = 1 + \tan^2 x.$$

$$\left.\begin{aligned} \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \end{aligned}\right\} \text{compound angle}$$

$$\left.\begin{aligned} \sin(2a) &= 2 \sin a \cos a \\ \cos(2a) &= \cos^2 a - \sin^2 a \end{aligned}\right\} \text{double angle.}$$

Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

Now let's try to differentiate $\sin x$.
As usual, we start by applying the definition.

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Now use compound angle (actually just about the only thing we can do)

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Split up

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

Factor

$$= \lim_{h \to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

Take out const factors.

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

Use our two limits.

$$= \sin x \cdot 0 + \cos x \cdot 1.$$

$$= \cos x.$$

Hence we have shown that

$$\boxed{\frac{d}{dx}(\sin x) = \cos x} \quad (x \text{ in radians})$$

(in particular $\sin x$ is diff everywhere)

Similarly we can show that $\cos x$ is diff everywhere and

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x.} \quad (x \text{ in radians}).$$

Ex.

a) $\frac{d}{dx}(2\sin(3\theta)) = 2\frac{d}{d\theta}(\sin(3\theta))$

$\underset{\text{rule}}{\overset{\text{chain}}{=}} 2 \cdot \cos(3\theta) \cdot \frac{d}{d\theta}(3\theta)$

$= 2\cos(3\theta) \cdot 3$

$= 6\cos(3\theta).$

b) $\frac{d}{dx}(\cos^2 x) \underset{\text{rule}}{\overset{\text{chain}}{=}} 2\cos x \frac{d}{dx}(\cos x)$

$= 2\cos x \cdot -\sin x$

$= -2\sin x \cos x$

$= -\sin 2x$ double angle.

c) $\frac{d}{dx}(\cos(x^2)) \underset{\text{rule}}{\overset{\text{chain}}{=}} -\sin(x^2)\frac{d}{dx}(x^2)$

$= -\sin(x^2) \cdot 2x$

$= -2x\sin(x^2).$

d) $\frac{d}{dt}\left(e^{-\sin t}\right) \underset{\text{rule}}{\overset{\text{chain}}{=}} e^{-\sin t} \cdot \frac{d}{dt}(-\sin t)$

$$= e^{-\sin t} \cdot -\cos t$$

$$= -\cos t \, e^{-\sin t}.$$

Other Trigonometric Functions

$\tan x = \frac{\sin x}{\cos x}$ suggests quotient rule

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \quad \text{as} \quad \sin^2 x + \cos^2 x = 1.$$

$$= \left(\frac{1}{\cos x}\right)^2$$

$$= \sec^2 x \quad \text{as} \quad \sec x = \frac{1}{\cos x} \text{ by defn.}$$

Hence

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x} \quad , x \text{ in radians.}$$

Similarly for $\cot x = \dfrac{\cos x}{\sin x}$

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x} \quad , x \text{ in radians.}$$

$\left(\operatorname{cosec} x = \dfrac{1}{\sin x}\right).$

Ex 5

a) $$\frac{d}{dt}(2\tan 3t) = 2\frac{d}{dt}(\tan 3t)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} 2 \cdot \sec^2 3t \cdot \frac{d}{dt}(3t)$$

$$= 2 \cdot \sec^2 3t \cdot 3t$$

$$= 6t \sec^2 3t$$

or $\dfrac{6t}{\cos^2 3t}$ (both ok).

b) $$\frac{d}{d\theta}(\cot(1-\theta))$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} -\operatorname{cosec}^2(1-\theta) \cdot \frac{d}{d\theta}(1-\theta)$$

$$= -\operatorname{cosec}^2(1-\theta) \cdot -1$$

$$= \operatorname{cosec}^2(1-\theta)$$

or $\dfrac{1}{\sin^2(1-\theta)}$

c) $\frac{d}{dt}\left(\frac{1+\tan t}{1-\tan t}\right)$

Use quotient rule

$$= \frac{(1-\tan t)\frac{d}{dt}(1+\tan t) - (1+\tan t)\frac{d}{dt}(1-\tan t)}{(1-\tan t)^2}$$

$$= \frac{(1-\tan t)\cdot \sec^2 t - (1+\tan t)\cdot -\sec^2 t}{(1-\tan t)^2}$$

$$= \frac{\sec^2 t - \cancel{\tan t \sec^2 t} + \sec^2 t + \cancel{\tan t \sec^2 t}}{(1-\tan t)^2}$$

$$= \frac{2\sec^2 t}{(1-\tan t)^2}$$

or

$$\frac{2}{\cos^2 t\,(1-\tan t)^2}$$

The remaining trig fns. $\sec x$, $\operatorname{cosec} x$.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{d}{dx}\left((\cos x)^{-1}\right)$$

power / chain rule

$$= -1 \cdot (\cos x)^{-2} \cdot \frac{d}{dx}(\cos x)$$

$$= \frac{-1}{\cos^2 x} \cdot -\sin x$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \tan x.$$

So

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad (x \text{ in rad}).$$

Similarly.

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

Ex.

$$\frac{d}{dx}(2\sec 4x) = 2\frac{d}{dx}(\sec 4x)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} 2 \cdot \sec 4x \tan 4x \frac{d}{dx}(4x)$$

$$= 2\sec 4x \tan 4x \cdot 4$$

$$= 8\sec 4x \tan 4x.$$

b) $\frac{d}{dx}\left(e^{\sec^2 x}\right)$

$\overset{\text{chain}}{\underset{\text{rule}}{=}} e^{\sec^2 x} \cdot \frac{d}{dx}(\sec^2 x)$

$\overset{\text{chain}}{\underset{\text{rule}}{=}} e^{\sec^2 x} \cdot 2\sec x \frac{d}{dx}(\sec x)$

$= e^{\sec^2 x} \cdot 2\sec x \cdot \sec x \tan x$

$= 2\sec^2 x \tan x \, e^{\sec^2 x}$.

Summary

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x.$$

c) $\frac{d}{dx}\left(\operatorname{cosec}\left(\sqrt{1+x^2}\right)\right)$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} -\operatorname{cosec}\left(\sqrt{1+x^2}\right)\cot\left(\sqrt{1+x^2}\right)\frac{d}{dx}\left(\sqrt{1+x^2}\right)$$

$$= -\operatorname{cosec}\left(\sqrt{1+x^2}\right)\cot\left(\sqrt{1+x^2}\right)\frac{d}{dx}\left((1+x^2)^{\frac{1}{2}}\right)$$

$$\overset{\text{chain}}{\underset{\text{rule}}{=}} -\operatorname{cosec}\left(\sqrt{1+x^2}\right)\cot\left(\sqrt{1+x^2}\right)\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot\frac{d}{dx}(1+x^2)$$

$$= -\operatorname{cosec}\left(\sqrt{1+x^2}\right)\cot\left(\sqrt{1+x^2}\right)\cdot\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x$$

$$= -\frac{x\operatorname{cosec}\left(\sqrt{1+x^2}\right)\cot\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}.$$