§ 3.4 The Chain Rule

Suppose we try to differentiate a composite function $f(g(x))$.

Let $z = g(x)$ and $y = f(z)$ so that $y = f(g(x))$.

Then a small change Δx in x produces a small change Δz in z which in turn produces a small change Δy in y. Provided $\Delta x, \Delta z \neq 0$, we can write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x}.$$

Since $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, this suggests that as $\Delta x, \Delta y, \Delta z$ get smaller and smaller, we have.

The Chain Rule (Leibniz Form)

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Note how the Leibniz form allows us to treat derivatives almost like fractions!

Since $\frac{dy}{dz} = f'(z)$ and $\frac{dz}{dx} = g'(x)$, we can also write

$$\frac{d}{dx}(f(g(x))) = f'(z) \cdot g'(x).$$

Substituting $z = g(x)$, we get

Theorem 3.5: The Chain Rule

If f and g are diff. f^{ns}, then

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

In words:

The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

Another way of writing this is to let $u = g(x)$ and then say

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

<u>Note:</u> This Leibniz form of the chain rule almost allows us to treat derivatives like fractions!

<u>Ex1.</u> The Length L in cm of a steel bar depends on the air temp H in °C which in turn depends on time t, measured in hours. If L incr. by 2cm for every degree incr in temp. and the temp is incr at 8°C/hour, how fast is the length of the bar incr? What are the correct units for your answer?

We expect the correct units to be in cm / hr.

Rate length incr wrt. temp. = $\frac{dL}{dh}$ = 2cm / °c

Rate temp incr wrt time = $\frac{dH}{dt}$ = 3 °c / hr.

We want $\frac{dL}{dt}$. By the chain rule

$$\frac{dL}{dt} = \frac{dL}{dH} \cdot \frac{dH}{dt}$$

$$= (2 \text{ cm}/°c)(3°c/hr)$$

$$= 6 \text{ cm}/hr.$$

Ex2. Differentiate each of the following.

a) $(x^2+1)^{100}$.

This can be written as $f(g(x))$
where $z = g(x) = x^2+1$ and $f(z) = z^{100}$.
$g'(x) = 2x$ $\qquad f'(z) = 100z^{99}$

Hence

$$\frac{df}{dx} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= 100z^{99} \cdot 2x$$

$$= 100(x^2+1)^{99} \cdot 2x$$

$$= 200x(x^2+1)^{99}$$

Always convert back to x at the end!

b) $\sqrt{3x^2+5x-2}$

Here $z = g(x) = 3x^2+5x-2$, $f(z) = \sqrt{z}$

so $g'(x) = 6x+5$, $f'(z) = \dfrac{1}{2\sqrt{z}}$

and

$$\frac{d}{dx}\left(\sqrt{3x^2+5x-2}\right) = f'(z) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{z}} \cdot (6x+5)$$

$$= \frac{6x+5}{2\sqrt{3x^2+5x-2}}.$$

c). $\dfrac{1}{x^2+x^4}$

Here $z = g(x) = x^2 + x^4$, $f(z) = \dfrac{1}{z}$

so $g'(x) = 2x + 4x^3$, $f'(z) = -\dfrac{1}{z^2}$

and by the chain rule

$$\frac{d}{dx}\left(\frac{1}{x^2+x^4}\right) = f'(g(x)) \cdot g'(x)$$

$$= \frac{-1}{(x^2+x^4)^2} \cdot 2x + 4x^3$$

$$= -\frac{2x+4x^3}{(x^2+x^4)^2}$$

$$= -2x\frac{1+2x^2}{(x^2+x^4)^2}$$

can also use quotient rule.

e) e^{x^2}.

Here $g(x) = x^2$, $f(z) = e^z$

$g'(x) = 2x$, $f'(z) = e^z$, so

$$\frac{d}{dx}(e^{x^2}) = f'(g(x)) \cdot g'(x)$$

$$= e^{x^2} \cdot 2x$$

$$= 2xe^{x^2}.$$

Examples a), b), c) were special cases of a general <u>power rule</u>

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x).$$

Pf. Let $g(x) = z$, $f(z) = z^n$

Then $f'(z) = nz^{n-1}$, so

$$\frac{d}{dx}(g(x))^n = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$= n(g(x))^{n-1} \cdot g'(x).$$

Ex3. $\sqrt{e^{-x/7}+5}$

This is of the form $\sqrt{g(x)}$
where $g(x) = e^{-x/7}+5$

To find $g'(x)$, need the chain rule.

Let $u = h(x) = -\frac{x}{7}$, $k(u) = e^{u}+5$

so that $g(x) = k(h(x))$.

Then $h'(x) = -\frac{1}{7}$, $k'(u) = e^{u}$

and so

$$
\begin{aligned}
g'(x) &= k'(u)\,h'(x) \\
&= e^{u} \cdot -\frac{1}{7} \\
&= -\frac{1}{7} \cdot e^{-x/7}.
\end{aligned}
$$

Then by the power rule.

$$\frac{d}{dx}\left(\sqrt{e^{-x/7}+5}\right) = \frac{d}{dx}\left(g(x)\right)^{\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{g(x)}} \cdot g'(x)$$

$$= \frac{1}{2\sqrt{e^{-x/7}+5}} \cdot -\frac{1}{7} e^{-x/7}$$

$$= -\frac{e^{-x/7}}{14\sqrt{e^{-x/7}+5}}.$$

b) $e^{(x+e^{x^2})}$.

Let $z = g(x) = x + e^{x^2}$, $f(z) = e^z$, $f'(z) = e^z$

so $e^{(x+e^{x^2})} = f(g(x))$.

Then

$$\frac{d}{dx}\left(e^{(x+e^{x^2})}\right) = f'(z) \cdot g'(x)$$
$$= e^z g'(x)$$
$$= e^{x+e^{x^2}} \cdot g'(x).$$

To find $g'(x)$, need chain rule again.

Let $u = h(x) = x^2$, $k(u) = e^u$

So $e^{x^2} = k(h(u))$.

$h'(x) = 2x$ and $k'(u) = e^u$, so

$$\frac{d}{dx}\left(e^{x^2}\right) = k'(u)\, h'(x)$$
$$= e^u \cdot 2x$$
$$= 2xe^{x^2}.$$

Hence

$$g'(x) = \frac{d}{dx}\left(x + e^{x^2}\right)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}\left(e^{x^2}\right)$$

$$= 1 + 2xe^{x^2}.$$

Putting all this together,

$$\frac{d}{dx}\left(e^{(x+e^{x^2})}\right) = e^{x+e^{x^2}} \cdot \left(1+2xe^{x^2}\right)$$

$$= \left(1+2xe^{x^2}\right)e^{x+e^{x^2}}.$$

Ex 4. Differentiate $\sqrt{1+e^{\sqrt{3+x^2}}}$

Chain rule is needed 4 times.

$$\frac{d}{dx}\left(\sqrt{1+e^{\sqrt{3+x^2}}}\right) = \frac{d}{dx}\left(1+e^{\sqrt{3+x^2}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(1+e^{\sqrt{3+x^2}}\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(1+e^{\sqrt{3+x^2}}\right)$$

$$= \frac{1}{2}\left(1+e^{\sqrt{3+x^2}}\right)^{-\frac{1}{2}} \cdot e^{\sqrt{3+x^2}} \frac{d}{dx}\left(\sqrt{3+x^2}\right)$$

$$= \frac{1}{2}\left(1+e^{\sqrt{3+x^2}}\right)^{-\frac{1}{2}} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{2}(3+x^2)^{-\frac{1}{2}} \frac{d}{dx}(3+x^2)$$

$$= \frac{1}{2}\left(1+e^{\sqrt{3+x^2}}\right)^{-\frac{1}{2}} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{\cancel{2}}(3+x^2)^{-\frac{1}{2}} \cdot \cancel{2}x$$

$$= \frac{x e^{\sqrt{3+x^2}}}{2\sqrt{(3+x^2)\left(1+e^{\sqrt{3+x^2}}\right)}}$$

Using the Product and Chain Rules to Differentiate a Quotient.

Ex 5. Find $k'(x) = \dfrac{x}{x^2+1}$.

Do this two different ways.

First; using the quotient rule.

$$k'(x) = \frac{\frac{d}{dx}(x) \cdot (x^2+1) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{1 \cdot (x^2+1) - x \cdot 2x}{(x^2+1)^2}$$

$$= \frac{1 - x^2}{(x^2+1)^2}.$$

The second way is to write

$$k(x) = x\,(x^2+1)^{-1}$$

and use the product rule

$$k'(x) = \frac{d}{dx}(x)\,(x^2+1)^{-1} + x\,\frac{d}{dx}\left((x^2+1)^{-1}\right)$$

$$= 1\,(x^2+1)^{-1} + x\,\frac{d}{dx}\left((x^2+1)^{-1}\right)$$

Now use the chain rule to diff $(x^2+1)^{-1}$. Let $z = g(x) = x^2+1$, $f(z) = \frac{1}{z}$

so $g'(x) = 2x$, $f'(z) = -\frac{1}{z^2}$

Then by chain rule.

$$\frac{d}{dx}\left((x^2+1)^{-1}\right) = f'(z) \cdot g'(x)$$

$$= -\frac{1}{z^2} \cdot 2x$$

$$= -\frac{1}{(x^2+1)^2} \cdot 2x.$$

Subst into $k'(x)$, we get.

$$k'(x) = (x^2+1)^{-1} + x \cdot \frac{-2x}{(x^2+1)^2}$$

$$= \frac{1}{x^2+1} - \frac{2x^2}{(x^2+1)^2}$$

$$= \frac{x^2+1}{(x^2+1)^2} - \frac{2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

which is the same answer as before.

Of course, this method was longer. In practice, just which method is quickest is a matter of experience.