MTH 436

Final Exam - Real Analysis II

Due Monday May 10th. at 3:00 PM in Lippit 200B

Name:

Solve 5 problems. Show all your work.!

1). Suppose that $f^{(n+1)}$ exists on (a,b), $x_0, x_1, \dots x_n$ are in (a,b), and p is the polynomial of degree $\leq n$ such that $p(x_i) = f(x_i)$, $0 \leq i \leq n$. Prove that if $x \in (a,b)$, then

$$f(x) \;=\; p(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

where c, which depends on x, is in (a,b).

2). Suppose that f is continuous and $F(x) \;=\; \int_a^x f(t)\,dt$ is bounded on $[a,\, b)$.

Suppose also that $g > 0$, g' is nonnegative and locally integrable on $[a,\, b)$, and $\lim_{x\to b^-} g(x) = \infty$. Show that

$$\lim_{x\to b^-} \frac{1}{[g(x)]^\rho} \int_a^x f(t)g(t)\,dt \;=\; 0. \qquad \rho > 1.$$

If in addition we assume that $\int_a^b f(t)dt$ converges. Show that

$$\lim_{x\to b^-} \frac{1}{g(x)} \int_a^x f(t)g(t)\,dt \;=\; 0.$$

Definition: Let f and g be defined on $[a,\, b]$. We say that f is *Riemann-Stieltjes integrable* with respect to g on $[a,\, b]$ if there is a number L with the following property: For every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\Big|\sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] - L\Big| < \epsilon,$$

provided only that $P \;=\; \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,\, b]$ such that $||P|| < \delta$ and $x_{j-1} \leq c_j \leq x_j$, $j = 1, 2, \dots, n$. In this case, we say that L is the *Riemann-Stieltjes integral* of f with respect to g over [a, ḃ], and write

$$\int_a^b f(x)\,dg(x) \;=\; L.$$

3). Choose ONE of the following:

3a). Suppose that f and g'' are bounded and fg' is integrable on $[a,b]$. Show that $\int_a^b f(x)\,dg(x)$ exists and equals $\int_a^b f(x)g'(x)\,dx$.

3b). Suppose that g' is integrable and f is continuous on $[a,b]$. Show that $\int_a^b f(x)\,dg(x)$ exists and equals $\int_a^b f(x)g'(x)\,dx$.

4). Choose ONE of the following:

4a). Suppose that $\{f_n\}$ converges pointwise on $[a,b]$ and, for each $x \in [a,b]$, there is an open interval I_x containing x such that $\{f_n\}$ converges uniformly on $I_x \cap [a,b]$. Show that $\{f_n\}$ converges uniformly on $[a,b]$.

4b). Show that if $\sum |a_n| < \infty$, then $\sum a_n \cos(nx)$ and $\sum a_n \sin(nx)$ define continuous functions on $(-\infty, \infty)$.

5). Choose ONE of the following:

5a.) Show that each point of the Cantor set $\mathbb{F}$ is a cluster point of $\mathbb{F}$.

5b.) Let $(K_n : n \in \mathbb{N})$ be a sequence of non-empty compact sets in $\mathbb{R}$ such that $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supseteq \cdots$. Prove that there exists at least one point $x \in \mathbb{R}$ such that $x \in K_n$ for all $n \in \mathbb{N}$; that is, the intersection $\cap_{n=1}^{\infty} K_n$ is not empty.