1) Let D be the Dirichlet function in $[0,1]$. To prove that D is not integrable in $[0,1]$ we use Th 5.10 text. Let $P=\{x_i : i=0,1,\dots,n\}$ be a partition of $[0,1]$. From the density of rationals and the density of irrationals, we obtain

$$\omega(D,[x_{i-1},x_i])=1 \text{ for } i=1,2,\dots,n.$$

Thus

$$\sum_{i=1}^{n}\omega(D,[x_{i-1},x_i])(x_i-x_{i-1})=\sum_{i=1}^{n}(x_i-x_{i-1})=1-0=1.$$

We conclude that the condition of Th 5.10 is not satisfied for any $\varepsilon<1$. Therefore, D is not integrable in $[0,1]$.

2) Let ${}^tP=\{(c_i,[x_{i-1},x_i]) : i=1,\dots,n\}$ be any tagged partition of $[0,1]$. From the definition of f we obtain

$$|S(f,{}^tP)|=\left|\sum_{i=1}^{n}f(c_i)(x_i-x_{i-1})\right|=|f(c_1)(x_1-x_0)|\le$$

$$\le |x_1-x_0|\le \|{}^tP\|. \qquad (1)$$

To prove that f is integrable in $[0,1]$ and $\int_0^1 f=0$, we use Def. 5.4 text. Let $\varepsilon>0$ be given. Let $\delta=\varepsilon$. Take any tagged partition ${}^tP$ such that $\|{}^tP\|<\delta$. By (1) we have

$$|S(f,{}^tP)-0|\le\|{}^tP\|<\delta=\varepsilon.$$

Thus f is integrable in $[0,1]$ and $\int_0^1 f=0$.

3) Let ${}^{t}P = \{(c_i, [x_{i-1}, x_i]) : i = 1, \ldots, n\}$ be any tagged partition of $[a, b]$. As $m \le f(x) \le M$ for all $x \in [a, b]$, we obtain

$$S(f, {}^{t}P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} M(x_i - x_{i-1}) = M(b-a). \quad (2)$$

Similarly

$$S(f, {}^{t}P) \geq m(b-a) \quad (3)$$

Let $\{{}^{t}P_n\}_{n=1}^{\infty}$ be a sequence of tagged partitions of $[a, b]$ such that $\lim_{n \to +\infty} \|{}^{t}P_n\| = 0$. By Th. 5.14 text,

$$S(f, {}^{t}P_n) \to \int_a^b f \quad \text{as } n \to +\infty.$$

Since $S(f, {}^{t}P_n)$ satisfies (2) and (3) for all $n = 1, 2, \ldots$ we conclude that

$$m(b-a) \le \int_a^b f \le M(b-a).$$

4) Note $\dfrac{f(1) - f(-1)}{g(1) - g(-1)} = \dfrac{0}{2} = 0$. Also for every $c \in (-1, 1)$,

$c \neq 0$:

$$\frac{f'(c)}{g'(c)} = \frac{2c}{3c^2} = \frac{2}{3c} \neq 0$$

For $c = 0$, the quotient $\dfrac{f'(c)}{g'(c)}$ is undefined. So there is no c for which

$$\frac{f'(c)}{g'(c)} = \frac{f(1) - f(-1)}{g(1) - g(-1)}. \quad (4)$$

The latter conclusion does not contradict the GMVT. The GMVT requires that for some $c \in (-1, 1)$

$$f'(c)\,(g(1)-g(-1)) = g'(c)\,(f(1)-f(-1)). \quad (5)$$

which holds with $c = 0$. The example shows that (4) and (5) are not equivalent.

5) Obsere that for any ${}^{t}P \in {}^{t}\mathcal{P}([a,b])$

$$S(f, {}^{t}P) = c(b-a).$$

Thus

$$\int_a^b f = c(b-a)$$

by Def. 5.4 text.