Perfect graphs I: Origins, a theorem, and a conjecture

Michael D. Barrus

Department of Mathematics
Brigham Young University

Graphs and Matrices Seminar
October 3, 2012
Definitions

$\chi$: chromatic number \hspace{1cm} $\omega$: clique number \hspace{1cm} $\chi \geq \omega$

$\theta$: clique cover number \hspace{1cm} $\alpha$: independence number \hspace{1cm} $\theta \geq \alpha$

$\chi$-perfect: Every induced subgraph satisfies $\chi = \omega$.

$\alpha$-perfect: Every induced subgraph satisfies $\alpha = \theta$.

CLASS 1: Graphs for which $\Theta = \alpha$ for every induced subgraph.

CLASS 2: Graphs for which $\alpha = \theta$ for every induced subgraph.

CLASS 3: Graphs for which $\chi = \omega$ for every induced subgraph.

CLASS 4: Graphs containing no induced odd cycles of length $\geq 5$ or their complements.
Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).
Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).

Chordal graphs are $\alpha$-perfect [Hajnal, Suranyi, 1959].
Chordal graphs

Graphs where every cycle of length at least 4 has a chord (i.e., graphs with no induced cycles of length at least 4).

Chordal graphs are $\alpha$-perfect [Hajnal, Suranyi, 1959].

Chordal graphs are $\chi$-perfect [Berge, 1960].
Chordal graphs are $\alpha$-perfect [Hajnal, Suranyi, 1959].

Chordal graphs are $\chi$-perfect [Berge, 1960].

Chordal graphs have simplicial orderings.
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)

Bipartite graphs are $\chi$-perfect.
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)

Bipartite graphs are $\chi$-perfect. They are also $\alpha$-perfect [König, 1916].
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)

Bipartite graphs are $\chi$-perfect. They are also $\alpha$-perfect [König, 1916].

Line graphs of bipartite graphs
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)

Bipartite graphs are $\chi$-perfect. They are also $\alpha$-perfect [König, 1916].

Line graphs of bipartite graphs

Line graphs of bipartite graphs are $\alpha$-perfect [König, Egerváry, 1931].
Bipartite graphs

Graphs with $\chi \leq 2$. (Equivalently, graphs with no odd cycles.)

Bipartite graphs are $\chi$-perfect. They are also $\alpha$-perfect [König, 1916].

Line graphs of bipartite graphs

Line graphs of bipartite graphs are $\alpha$-perfect [König, Egerváry, 1931].

They are also $\chi$-perfect [König, 1916].
Comparability graphs

Graphs modeling relationships in a poset.

![Diagram of comparability graph]

- Comparability graphs represent the relationships in a partially ordered set (poset).

Variables:
- a, b, c, d, e, f

M. D. Barrus (BYU)
Perfect Graphs I
Oct. 3, 2012
Comparability graphs

Graphs modeling relationships in a poset.

Comparability graphs are $\chi$-perfect [Mirsky, 1971].
Comparability graphs

Graphs modeling relationships in a poset.

Comparability graphs are $\chi$-perfect [Mirsky, 1971].

Comparability graphs are $\alpha$-perfect [Dilworth, 1950].
Berge’s conjectures (early 1960’s)

The Weak Perfect Graph Conjecture

Class 2 = Class 3.

In other words, a graph is $\chi$-perfect [$\alpha$-perfect] if and only if its complement is.

The Strong Perfect Graph Conjecture

Classes 2 and 3 are the same as Class 4.

In other words, the $\chi$-perfect [$\alpha$-perfect] graphs are exactly those graphs having no induced odd hole or odd antihole.

Berge also conjectured that Class 4 $\subseteq$ Class 1.
The (Weak) Perfect Graph Theorem

Theorem (Lovász, 1972)
A graph is perfect if and only if $\omega(H)\alpha(H) \geq |V(H)|$ for every induced subgraph $H$.

Corollary (Perfect Graph Theorem)
A graph $G$ is perfect if and only if $\overline{G}$ is perfect.
Proof sketch (Gasparyan, 1996)

Note that $\omega(H)\alpha(H) \geq |V(H)|$ for induced subgraphs of $\chi$-perfect graphs.
Proof sketch (Gasparyan, 1996)

Note that $\omega(H) \alpha(H) \geq |V(H)|$ for induced subgraphs of $\chi$-perfect graphs.

Approach: Show that $\omega(H) \alpha(H) < |V(H)|$ for some $H$ in every imperfect graph.
Proof sketch (Gasparyan, 1996)

Note that $\omega(H)\alpha(H) \geq |V(H)|$ for induced subgraphs of $\chi$-perfect graphs.

Approach: Show that $\omega(H)\alpha(H) < |V(H)|$ for some $H$ in every imperfect graph.

It suffices to consider p-critical subgraphs (minimal imperfect subgraphs.)

\[ C_7 \]
Proof sketch, continued

Let \( a = \alpha(G) \) and \( w = \omega(G) \). Let \( S_0 = \{x_1, \ldots, x_a\} \) be a maximum independent set in \( G \).
Proof sketch, continued

Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in $G$.

**Lemma**: Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence $G - x_r$ is $w$-colorable, and the color classes partition the rest of the vertices into sets $S_i$. 
Proof sketch, continued

Let $a = \alpha(G)$ and $w = \omega(G)$. Let $S_0 = \{x_1, \ldots, x_a\}$ be a maximum independent set in $G$.

**Lemma:** Deleting any independent set in a $p$-critical graph leaves the clique number unchanged.

Hence $G - x_r$ is $w$-colorable, and the color classes partition the rest of the vertices into sets $S_i$

$S_0 = \{x_1, x_2\}$, $S_1 = \{a, d\}$, $S_2 = \{b, e\}$, $S_3 = \{x_2, f\}$
Proof sketch, continued

Let \( a = \alpha(G) \) and \( w = \omega(G) \). Let \( S_0 = \{x_1, \ldots, x_a\} \) be a maximum independent set in \( G \).

**Lemma:** Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence \( G - x_r \) is \( w \)-colorable, and the color classes partition the rest of the vertices into sets \( S_i \):

\[
S_0 = \{x_1, x_2\}, \quad S_1 = \{a, d\}, \quad S_2 = \{b, e\}, \quad S_3 = \{x_2, f\},
\]
\[
S_4 = \{a, e\}, \quad S_5 = \{b, f\}, \quad S_6 = \{x_1, d\}
\]

\( aw + 1 \) (overlapping) sets in all
Proof sketch, continued

\[ S_1 = \{a, d\}, \quad Q_1 = \{x_1, e, f\} \]

Each \( S_i \) is an independent set.

**Lemma:** Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence \( G - S_i \) contains a maximum clique \( Q_i \) that is maximum in \( G \).
Proof sketch, continued

\[ S_1 = \{a, d\}, \quad Q_1 = \{x_1, e, f\} \]

Each \( S_i \) is an independent set.

**Lemma:** Deleting any independent set in a p-critical graph leaves the clique number unchanged.

Hence \( G - S_i \) contains a maximum clique \( Q_i \) that is maximum in \( G \).

**Lemma:** Each \( Q_i \) intersects \( S_j \) when \( i \neq j \).
Proof sketch, concluded

Let $A$ and $B$ be $|V(G)|$-by-$(aw + 1)$ incidence matrices for the families $S_i$ and $Q_i$.

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}$$
Proof sketch, concluded

Let $A$ and $B$ be $|V(G)|$-by-$(aw + 1)$ incidence matrices for the families $S_i$ and $Q_i$.

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\end{bmatrix}$$

Since $|S_i \cap Q_j| = 1$ when $i \neq j$, and $|S_i \cap Q_i| = 0$, we have

$$A^T B = J - I.$$
Proof sketch, concluded

Let $A$ and $B$ be $|V(G)|$-by-$(aw + 1)$ incidence matrices for the families $S_i$ and $Q_i$.

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

Since $|S_i \cap Q_j| = 1$ when $i \neq j$, and $|S_i \cap Q_i| = 0$, we have

$$A^T B = J - I.$$ 

Rank $J - I = aw + 1$, so $A$ and $B$ have rank at least $aw + 1$. Thus $|V(G)| \geq aw + 1$ and hence $aw < |V(G)|$. □
**The (Weak) Perfect Graph Theorem**

Class 2 = Class 3.

*In other words, a graph is $\chi$-perfect [\(\alpha\)-perfect] if and only if its complement is.*

**The Strong Perfect Graph Conjecture**

Classes 2 and 3 are the same as Class 4.

*In other words, the $\chi$-perfect [\(\alpha\)-perfect] graphs are exactly those graphs having no induced odd hole or odd antihole.*