§ 4.5 Moment-Generating Functions

A shortcut to calculating the moments of a distribution.

**Defn** The moment-generating function (MGF) $M_X(t)$ of a random variable $X$ is given by

$$M_X(t) = E(e^{tx})$$

When $X$ is discrete, we have (by Thm 4.1)

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} \cdot f(x).$$

When $X$ is cts, we have (by Thm 1.1 again)

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \, dx.$$
As an object, $M_X(t)$ is a function from $\mathbb{R} \to \mathbb{R}$ given by the rule
\[ t \mapsto E(e^{tx}). \]

Why the name moment-generating function?

Recall that $e^{tx}$ has Maclaurin series expansion
\[ e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^nx^n}{n!} + \cdots. \]

In the discrete case, we get
\[ M_X(t) = E(e^{tx}) = \sum_{x} e^{tx} \cdot f(x) = \sum_{x} \left( 1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^nx^n}{n!} \right) \cdot f(x). \]
\[ = \sum_x f(x) + t \sum_x x f(x) + \frac{t^2}{2!} \sum_x x^2 f(x) + \ldots \]

\[ = 1 + \mu t + \mu_2 \frac{t^2}{2!} + \ldots + \mu_r \frac{t^r}{r!} + \ldots \]

( formal interchange of sums ok )

Thus the \[ M_x(t) \] itself has a Maclaurin series about 0 and the coefficient of \[ t^r \] in this series is \[ \frac{\mu_r}{r!} \] (where \( \mu_r \) is the \( r \)-th moment of \( X \) about 0).

A similar argument holds in the continuous case.
Recall that if \( f \) for \( g(t) \) has
McLaurin series
\[
g(t) = \sum_{r=0}^{\infty} a_r \frac{t^r}{r!}
\]
about 0, then
\[
a_r = \frac{d^r g(t)}{dt^r} \bigg|_{t=0}.
\]

Applying this to the MGF of \( X \) gives,

Theorem 4.0.9

\[
\frac{d^r M_X(t)}{dt^r} \bigg|_{t=0} = \mu_r,
\]

This result shows us how to obtain
the moments of \( X \) from the MGF.
Ex. Find the MGF of the r.v. whose pdf is

$$f(x) = \begin{cases} 
  e^{-x}, & x > 0 \\
  0, & x \leq 0
\end{cases}$$

and use it to find an expression for $\mu_k$.

Soln. By defn

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) \, dx$$

$$= \int_{0}^{\infty} e^{tx} e^{-x} \, dx$$

$$= \int_{0}^{\infty} e^{-(1-t)x} \, dx$$

$$= \frac{1}{1-t} \quad \text{for } t < 1$$

$$\left( \int_{0}^{\infty} e^{-ax} \, dx = \frac{1}{a} \right)$$

for $a > 0$
It is well known that

\[ \frac{1}{1-t} \]

has Maclaurin series

\[ \frac{1}{1-t} = 1 + t + t^2 + t^3 + \ldots + t^r + \ldots \quad \text{if} \ 1 \mid t \mid < 1 \]

Thus

\[ M_X(t) = 1 + t + t^2 + t^3 + \ldots + t^r + \ldots \]

\[ = 1 + 1! \frac{t}{1!} + 2! \frac{t^2}{2!} + \ldots + r! \frac{t^r}{r!} + \ldots \]

and so \( \mu_r = r! \), \( r = 0, 1, 2, \ldots \)

Often the main difficulty with using MGFs is not finding \( M_X(t) \), but finding its Maclaurin series. However, if we only require a few of the moments, this problem can be got round by appealing to Thm 4.9 and differentiating.
Ex. Suppose $X$ is discrete with pdf

\[ f(x) = \frac{1}{8} \binom{3}{x}, \quad x = 0, 1, 2, 3. \]

Find the MGF of $X$ and use it to determine $\mu_1'$ and $\mu_2'$.

\[ M_X(t) = E(e^{tx}) = \frac{1}{8} \sum_{x=0}^{3} e^{tx} \binom{3}{x} \]

\[ = \frac{1}{8} \left( 1 + 3e^t + 3e^{2t} + e^{3t} \right) \]

\[ = \frac{1}{8} \left( 1 + e^t \right)^3 \]

It would be rather hard to find the Maclaurin series of this fn. However, by Thm 4.9

\[ \mu_1' = M_X'(0) = \frac{3}{8} (1+e^t)^2; e^t \bigg|_{t=0} = \frac{3}{2} \]
and

\[
M_2' = M_X''(0) = \frac{3}{4} (1 + e^t) e^{2t} \\
+ \frac{3}{8} (1 + e^t)^2 e^t \bigg|_{t=0} \\
= \frac{3}{4} \cdot 2 + \frac{3}{8} \cdot 4 \\
= 3.
\]

Finally, a useful result for calculating MGFs

Thm 4.10 If \( a \) and \( b \) are constants, then

1. \( M_{X+a}(t) = E(e^{(X+a)t}) = e^{at} M_X(t) \)

2. \( M_{bX}(t) = E(e^{bXt}) = M_X(bt) \)

3. \( M_{\frac{X+a}{b}}(t) = E(e^{\left(\frac{X+a}{b}\right)t}) = e^{\frac{at}{b}} M_X\left(\frac{t}{b}\right) \)

Pf. Exercise. (easy)
Part 1 of this thm is useful when $a = -\mu$
and part 2 is useful when $a = -\mu$, $b = \sigma$
in which case

$$M_{X-\mu}(t) = e^{\frac{\mu t}{\sigma}} M_{X}(\frac{t}{\sigma})$$

This trick is useful for using MGFs
to 'prove' the central limit theorem
(see later).

As an exercise, show that if $X$
has mean $\mu$ and variance $\sigma$, then

$$\frac{X-\mu}{\sigma}$$

has mean 0 and variance 1.