We may also consider the case of two r.v.'s.

**Defn.** A bivariate f.d. with values \( f(x,y) \) defined on \( \mathbb{R}^2 \) is called a **joint probability density function** of the r.v.'s \( X \) and \( Y \) iff

\[
P((x, y) \in A) = \int_A f(x,y) \, dx \, dy
\]

for any region \( A \) in the xy-plane.

Analogous to Thm 3.5 for one r.v., we have

**Theorem 3.8** A bivariate f.d. can serve as a joint pdf of a pair of r.v.'s \( X, Y \) if its values \( f(x,y) \) satisfy

1. \( f(x,y) \geq 0 \), \(-\infty < x, y < \infty\)
2. \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1 \).
Ex. Given the joint pdf

\[ f(x, y) = \begin{cases} \frac{3}{5} x(y + x), & 0 < x < 1, \ 0 < y < 2 \\ 0, & \text{elsewhere} \end{cases} \]

of two r.v.'s \(X, Y\), find \( P((X, Y) \in A) \)

where \(A\) is the region

\[ A = \{ (x, y) : 0 < x < \frac{1}{2}, \ 1 < y < 2 \} . \]

So.

\[ P((X, Y) \in A) = P(0 < x < \frac{1}{2}, \ 1 < y < 2) \]

\[ = \int_1^2 \int_0^{\frac{1}{2}} \frac{3}{5} x(y + x) \ dx \ dy \]

\[ = \int_1^2 \left[ \frac{3x^2y + 3x^3}{10} \right]_{x=0}^{x=\frac{1}{2}} \ dy \]

\[ = \int_1^2 \left[ \frac{3y^2}{40} + \frac{1}{40} \right] \ dy = \left[ \frac{3y^2}{80} + \frac{y}{40} \right]_1^2 = \frac{11}{80} . \]
We can also define a joint CDF for two r.v.'s.

**Defn.** If $X, Y$ are r.v.'s, the function given by

$$F(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t) \, ds \, dt,$$

$\quad -\infty < x, y < \infty$

Where $f(x, y)$ is the value of a joint pdf of $X, Y$ is called the joint cumulative distribution function of $X$ and $Y$.

Similarly to the univariate case.

**Thm.** If $F(x, y)$ is the value of the joint CDF of two r.v.'s $X$ & $Y$ at $(x, y)$, then

a) $\lim_{x \to -\infty, y \to -\infty} F(x, y) = 0$

b) $\lim_{x \to \infty, y \to \infty} F(x, y) = 1$

c) if $a < b$ and $c < d$, then $F(a, c) \leq F(b, d)$. 
Similarly to \( f(x) = \frac{dF(x)}{dx} \) for one cts. rv., for two cts. rvs. we have

\[
    f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}
\]

whenever these partial derivatives exist.

So as before

\[
    \text{Joint CDF} \quad \frac{\partial^2}{\partial x \partial y} \quad \text{Joint pdf.}
\]

As in §3.4, we let \( f(x,y) = 0 \) whenever the above relationship doesn't hold.
Ex. If the joint pdf of $X$ & $Y$ is given by

$$f(x,y) = \begin{cases} 
x+y, & 0 < x, y < 1 \\
0, & \text{elsewhere}
\end{cases}$$

find the joint CDF of these two r.v.'s.

**Solu.** If either $x < 0$ or $y < 0$, then $F(x,y) = 0$.

e.g. If $x < 0$

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t) \, ds \, dt$$

$$= \int_{x}^{y} 0 \, dt \quad \text{as} \quad f(s,t) = 0 \quad \text{as} \quad s \leq x < 0$$

$$= 0$$

(Region 0)
If $0 < x < 1$, $0 < y < 1$ (Region I),

$$F(x, y) = \int_0^x \int_0^y (s+t) \, ds \, dt = \frac{1}{2} xy (x+y).$$

If $1 < x$, $0 < y < 1$ (Region II),

$$F(x, y) = \int_0^1 \int_0^y (s+t) \, ds \, dt = \frac{1}{2} y (y+1).$$

If $0 < x < 1$, $1 < y$ (Region III),

$$F(x, y) = \int_0^1 \int_0^1 (s+t) \, ds \, dt = \frac{1}{2} x (x+1).$$

If $1 < x$, $1 < y$ (Region IV),

$$F(x, y) = \int_0^1 \int_0^1 (s+t) \, ds \, dt = 1.$$
\[ F(x, y) = \begin{cases} 
0, & x \leq 0, y \leq 0 \\
\frac{1}{2} xy(x+y), & 0 < x < 1, 0 < y < 1 \\
\frac{1}{2} y(y+1), & 1 < x, 0 < y < 1 \\
\frac{1}{2} x(x+1), & 0 < x < 1, 1 < y \\
1, & 1 \leq x, 1 \leq y. 
\end{cases} \]

N.b. Since \( F \) is continuous everywhere, it doesn't matter just which region we say the boundary lines (e.g., \( x = 1 \)) belong to.
Ex. Find the joint pdf of the two r.v.'s $X$ and $Y$ whose joint CDF is given by

$$F(x, y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Also use the joint pdf to find

$$P(1 < x < 3, 1 < y < 2).$$

Sln:

Partial differentiation gives

$$\frac{\partial^2 F}{\partial x \partial y} = e^{-(x+y)} \quad \text{if} \quad x, y > 0$$

and

$$\frac{\partial^2 F}{\partial x \partial y} = 0 \quad \text{if} \quad x < 0 \text{ or } y < 0.$$

We can then set $f(x, y) = 0$ on the remaining pts $(x=0 \text{ or } y=0)$ to get
e.g. For $P(1 < X < 3, 1 < Y < 2)$, the picture looks like:

![Graph showing the region where 1 < X < 3 and 1 < Y < 2]

All the defns in this section can be generalized to the multivariate case where we have n r.v.'s.

The joint pdf of n discrete r.v.'s is given by:

$$f(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

for each n-tuple $(x_1, \ldots, x_n)$ within the range of the r.v.'s.
\[ f(x, y) = \begin{cases} 
\frac{e^{-(x+y)}}{xy}, & x > 0, y > 0 \\
0, & \text{otherwise.}
\end{cases} \]

Then, by integration,

\[ P(1 < x < 3, 1 < y < 2) = \int_{1}^{2} \int_{1}^{3} e^{-(x+y)} \, dx \, dy = (e^{-1} - e^{-3})(e^{-1} - e^{-2}) \]

\[ = e^{-2} - e^{-3} - e^{-4} + e^{-5} \approx 0.074 \]

In terms of multivariable calculus, we can think of \( f(x, y) \) as a surface and probability corresponds to finding the volume under the surface on a certain region.
The joint CDF is given by

\[ F(x_1, x_2, \ldots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \]

for \(-\infty < x_1, \ldots, x_n < \infty\).

Ex. If the joint pdf of 3 discrete r.v.'s is given by

\[ f(x, y, z) = \frac{(x+y)^2}{63}, \quad \begin{array}{c} x=1,2,3 \vspace{2mm} \\ y=1,2,3 \vspace{2mm} \\ z=1,2 \end{array} \]

find \( P(X=2, Y+Z \leq 3) \).

\[ \sum_{y=1}^{3} \sum_{z=1}^{2} f(2, y, z) = f(2, 1, 1) + f(2, 1, 2) + f(2, 2, 1) \]

\[ = \frac{3}{63} + \frac{6}{63} + \frac{4}{63} \]

\[ = \frac{13}{63} \]
In the chi case, probs. are again obtained by integrating a pdf and
\[ P((X_1, \ldots, X_n) \in A) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n \]
for any region A \subset \mathbb{R}^n.

The joint CDF is given by
\[ F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(t_1, \ldots, t_n) \, dt_1 \cdots dt_n, \]
\[-\infty < x_1, \ldots, x_n < \infty.

Also partial diff gives
\[ f(x_1, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F(x_1, x_2, \ldots, x_n) \]
whenever these partial derivatives exist.
Ex. If the trivariate prob. density of $X_1, X_2, X_3$ is given by

$$f(x_1, x_2, x_3) = \begin{cases} (x_1 + x_2) e^{-x_3}, & 0 < x_1 < 1, 0 < x_2 < 1, 0 < x_3 \\ 0 & \text{otherwise} \end{cases}$$

Find $P((X_1, X_2, X_3) \in A)$ where

$$A = \left\{ (x_1, x_2, x_3) : 0 < x_1 < \frac{1}{2}, \frac{1}{2} < x_2 < 1, x_3 < 1 \right\}$$

SOLN.

$$P((X_1, X_2, X_3) \in A) = P(0 < x_1 < \frac{1}{2}, \frac{1}{2} < x_2 < 1, x_3 < 1)$$

$$= \int_{0}^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \int_{0}^{x_2} (x_1 + x_2) e^{-x_3} \, dx_1 \, dx_2 \, dx_3$$

$$= \int_{0}^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \left( \frac{1}{8} + \frac{x_2}{2} \right) e^{-x_3} \, dx_2 \, dx_3$$

$$= \int_{0}^{\frac{1}{2}} \frac{1}{4} e^{-x_3} \, dx_3 = \frac{1}{4} (1 - e^{-\frac{1}{2}}) \approx 0.158.$$