§ 2.7 Independent Events

Informally, two events are independent if the occurrence or non-occurrence of either one does not influence the probability of the other. Formally we have

**Defn 2.2** Two events $A$ and $B$ are independent iff

$$P(A \cap B) = P(A) \cdot P(B).$$

Note that by Theorem 2.9, if $P(A) \neq 0$, then

$$P(A \cap B) = P(A)P(B \mid A),$$

$$P(A) \cdot P(B) = P(A)P(B \mid A)$$

so

$$P(B) = P(B \mid A) \quad (0 \leq P(A) \neq 0).$$

Similarly, if $P(B) \neq 0$, then

$$P(A) = P(A \mid B).$$

If two events $A$ and $B$ are not independent, we say they are dependent.
Ex. Let $A, B$ be two events and suppose $P(A) = 0$. Then $A$ and $B$ are independent.

To see this note first that $A \cap B \subseteq A$, so $0 \leq P(A \cap B) \leq P(A) = 0$, by Postulate 1 & Thm 2.5 and so $P(A \cap B) = 0$.

Thus $P(A \cap B) = 0 = 0 \cdot P(B) = P(A) \cdot P(B)$.

Similarly, if $P(B) = 0$, then $A$ and $B$ are automatically independent.
Ex. Coin is tossed 3 times and each of the 8 possible outcomes has prob. $\frac{1}{8}$.

Let

$$A = \{ \text{head occurs on each of the} \}$$
$$\quad \quad \text{first two tosses}$$

$$B = \{ \text{tail occurs on third toss} \}$$

$$C = \{ \text{exactly 2 tails occur in the 3 tosses} \}$$

Show

a) $A$ & $B$ are indep.

b) $B$ & $C$ are dependent.

Solv. $A = \{ HHH, HHT \}$

$B = \{ HHT, HTT, THT, TTT \}$

$C = \{ HTT, THT, TTH \}$

$A \cap B = \{ HHT \}$

$B \cap C = \{ HTT, THT \}.$
\[ P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{3}{8}, \]

\[ P(A \cap B) = \frac{1}{8}, \quad P(B \cap C) = \frac{1}{4} \]

a) \[ P(A \cap B) = \frac{1}{8} \]

\[ P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}, \]

so \( A, B \) are indep.

b) \[ P(B \cap C) = \frac{1}{4} \]

\[ P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16} \neq \frac{1}{4} \]

so \( B, C \) are dep.
Thm 2.11  If \( A, B \) are indemp., then 
\( A, B^c \) are indemp.

Lpf. Since \( A = (A \cap B) \cup (A \cap B^c) \) and this union is disjoint

\[
P(A) = P((A \cap B) \cup (A \cap B^c))
\]
\[
= P(A \cap B) + P(A \cap B^c)
\]

Thus

\[
P(A \cap B^c) = P(A) - P(A \cap B)
\]
\[
= P(A) - P(A) \cdot P(B) \quad \text{by ind. of } A, B
\]
\[
= P(A)(1 - P(B))
\]
\[
= P(A)P(B^c) \quad \text{as } P(B^c) = 1 - P(B).
\]
\square
Independence for more than two events.

Defn 2.3 Events $A_1, A_2, \ldots, A_k$ are independent if the probability of the intersection of any $2, 3, \ldots, k$ events is the product of their respective probabilities.

i.e. for each $2 \leq i \leq k$, and each $1 \leq n_1 < n_2 < n_3 < \ldots < n_i \leq k$

$$P(\bigcap_{j=1}^{i} A_{n_j}) = \prod_{j=1}^{i} P(A_{n_j})$$

As the next example shows, 3 or more events can be pairwise independent without actually being independent.
Ex. Consider the following Venn diagram with probabilities assigned to the various regions.

From the diagram:
- \( P(A) = P(B) = P(C) = \frac{1}{4} \)
- \( P(A \cap B) = P(B \cap C) = P(A \cap C) = \frac{1}{4} \)
- \( P(A \cap B \cap C) = \frac{1}{4} \)

Thus,
- \( P(A) \cdot P(B) = \frac{1}{4} = P(A \cap B) \)
- \( P(A) \cdot P(C) = \frac{1}{4} = P(A \cap C) \)
- \( P(B) \cdot P(C) = \frac{1}{4} = P(B \cap C) \)

but
- \( P(A) \cdot P(B) \cdot P(C) = \frac{1}{8} \neq \frac{1}{4} = P(A \cap B \cap C) \).

Can also happen that \( P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C) \) but \( A, B, C \) not independent (see homework).
Ex. Find the probs of getting.

a) 3 heads in 3 random tosses of a balanced coin

b) four sires and then another number (ie not a six) in 5 random rolls of a balanced die.

Soh: Multiply probabilities to get

a) \[ \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \]

b) \[ \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{7,776} \]