2.4 The Postulates of Probability and the Probability of an Event

For a sample space $S$ and an event $B$, let $P(B)$ denote the probability of $B$. We have the following 3 postulates.

1. $P(B) \geq 0$ for all events $B \subseteq S$.

2. $P(S) = 1$.

3. If $A_1, A_2, \ldots$ is a finite or countably infinite sequence of pairwise disjoint (mutually exclusive) events, then

$$P\left( \bigcup A_i \right) = \sum_{i} P(A_i).$$

or

$$P\left( A_1 \cup A_2 \cup \ldots \right) = P(A_1) + P(A_2) + \ldots$$
Ex. An experiment has 4 possible outcomes A, B, C, D which are mutually exclusive. Explain why the following assignments of probability are not permissible.

a) \( P(A) = 0.12, \; P(B) = 0.63, \; P(C) = 0.45, \; P(D) = -0.2 \)

b) \( P(A) = \frac{1}{4}, \; P(B) = \frac{1}{3}, \; P(C) = \frac{1}{3}, \; P(D) = \frac{1}{4} \).

Thm 2.1 If \( A \) is an event in a discrete sample space, then \( P(A) \) equals the sum of the probabilities of the individual outcomes comprising \( A \).

Note: A discrete sample space always contains countably many points (technical).

Pf. Let \( O_1, O_2, O_3, \ldots \) be the outcomes (sample pts.) which comprise \( A \). Then by postulate 3, since the \( O_i \)'s are pairwise disjoint,

\[
P(A) = P\left( \bigcup_{i} O_i \right) = \sum_{i} P(O_i).
\]
Ex. A die is biased so that each odd number is twice as likely to appear as each even number. Find \( P(G) \) where \( G \) is the event that a number greater than 3 occurs on a single roll of the die.

Sthn.

\[ S = \{1, 2, 3, 4, 5, 6\}. \]

If each even no. has prob \( w \) & each odd no. then has prob \( 2w \), we have

\[ 2w + w + 2w + w + 2w + w = 1 \]

\[ (proportion 2) \]

\[ 9w = 1 \]

\[ w = \frac{1}{9} \]

So 2, 4, 6 each have prob \( \frac{1}{9} \) and 1, 3, 5 \( \frac{2}{9} \).
Now \[ G = \{4, 5, 6\} \]

and by Thm 2-1

\[
P(G) = P(4) + P(5) + P(6)
\]

\[
= \frac{1}{9} + \frac{2}{9} + \frac{1}{9}
\]

\[
= \frac{4}{9}.
\]
Ex. Flip a balanced coin twice.
What is the prob. of getting at least one head?

Sample space: \( S = \{ \text{HH, HT, TH, TT} \} \).

Coin is balanced, so each outcome has prob. \( \frac{1}{4} \).

Event that I get at least one head is given by

\[ A = \{ \text{HH, HT, TH} \} \]

and by the previous then

\[
P(A) = P(\text{HH}) + P(\text{HT}) + P(\text{TH})
= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}
= \frac{3}{4}
\]
Ex. If, for a given experiment \( O_1, O_2, O_3, \ldots \) is a countably infinite sequence of (distinct) outcomes verify that

\[
P(O_i) = \left( \frac{1}{2} \right)^i \quad \text{for } i = 1, 2, 3, \ldots
\]

gives a probability measure.

Postulates 1, 3 follow from Thm 2.1, so we just need to check postulate 2, i.e. \( P(S) = 1 \).

But \( P(S) = P(\bigcup_{i=1}^\infty O_i) = \sum_{i=1}^\infty P(O_i) \) by Thm 2.1

\[
= \sum_{i=1}^\infty \left( \frac{1}{2} \right)^i
\]

\[
= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.
\]
Thm 2.2  If an experiment can result in any one of \( N \) equally likely distinct outcomes and if \( n \) of these outcomes constitute an event \( A \), then

\[
P(A) = \frac{n}{N} \quad \left( \frac{\# \text{ favourable}}{\# \text{ possible}} \right).
\]

Pf. Let \( O_1, \ldots, O_n \) be the outcomes which comprise \( A \). Then \( A = O_1 \cup \cdots \cup O_n \) and \( P(O_i) = \frac{1}{N} \) for each \( 1 \leq i \leq n \), so

\[
P(A) = P(O_1 \cup \cdots \cup O_n)
\]

\[
= \sum_{i=1}^{n} P(O_i)
\]

\[
= \sum_{i=1}^{n} \frac{1}{N}
\]

\[
= \frac{n}{N}.
\]

\( \square \)
Ex. What is the prob. of getting a full house (3 of a kind and a pair) in a 5-card poker hand dealt from a deck of 52 cards?

Soln. For a particular full house,
(e.g. 3 kings, 2 aces) there are

\[
\binom{4}{3} \binom{4}{2}
\]

ways this can happen, while there are

\[13 \times 12\]

possible types of card for a full house.

The total number of full houses is then

\[13 \times 12 \binom{4}{3} \binom{4}{2}\]

while the total number

of poker hands is \(\binom{52}{5}\).

The probability of a full house is then by Thm 2.2

\[
\frac{\text{# favourable}}{\text{# possible}} = \frac{13 \times 12 \binom{4}{3} \binom{4}{2}}{\binom{52}{5}} = 0.0014,
\]