Z-Cyclic Generalized Whist Frames and Z-Cyclic Generalized Whist Tournaments

Norman J. Finizio and Brian J. Travers
Department of Mathematics
University of Rhode Island
Kingston, RI 02881
finizio@uriacc.uri.edu; travers@math.uri.edu

Abstract

Much of the work in this article was inspired by the elegant and powerful method introduced by G. Ge and L. Zhu in their recent paper on triplewhist frames. We extend their ideas to generalized whist tournament designs. Thus, in one sense, we provide a complete generalization of their methodology. We also incorporate the product theorems of Anderson et al. to broaden their class of Z-cyclic frames. Our techniques are illustrated by the production of many new Z-cyclic $(2,6)$ GWhD(v) that would be difficult to produce by any other existing method.

1 Introduction

In [14] G. Ge and L. Zhu introduced Z-cyclic triplewhist frames and showed how they were useful for the construction of Z-cyclic triplewhist tournaments. Their methodology is both elegant and efficient. In particular, an application of their techniques greatly simplified the problem of constructing Z-cyclic triplewhist tournaments when the number of players is of the form $3qp$ where

$q \equiv 3 \pmod 4$ and $p \in \{5, 13, 17\}$. Here we extend and generalize their work to the case of generalized whist tournament designs. Such designs are a relatively new combinatorial object, having been introduced by Abel et al. in [2].

Definition 1.1 *Let t, e, k, v be integers such that k=et and $v \equiv 0, 1 (mod\ k)$. A (t, k) Generalized Whist Tournament on v players, denoted $(t, k)GWhD(v)$, is a (near) resolvable $(v, k, k-1)BIBD$ having the following properties : (i) each block is considered to be a game involving e teams of t players each, and (ii) every pair of players, say x and y, appear together in the same game exactly $t - 1$ times as teammates and exactly $k - t$ times as opponents.*

The (near) resolution classes of a (t, k)GWhD(v) are called rounds of the design. A game is typically written in the form $(a_{11}, a_{12}, \ldots, a_{1t}; a_{21}, a_{22}, \ldots, a_{2t}; \ldots; a_{e1}, a_{e2}, \ldots, a_{et})$, where the semicolons delineate the teams.

It is to be noted that the standard whist tournament design corresponds to the choice $(t, k) = (2, 4)$. The literature for the whist tournament problem is extensive; see [7, 8, 9]. Whist tournaments on v players are usually denoted $Wh(v)$. For some literature related to the specific cases $(t, k) = (2, 6), (3, 6)$, and $(4, 8)$ see [1, 3, 4, 5]. Throughout this study we make considerable implicit use of tournament designs and frames. Excellent sources of information regarding these concepts are [6] and [13], respectively.

Definition 1.2 *A frame is a group divisible design $GDD_\lambda(X,\mathcal{G},\mathcal{B})$, such that (i) the size of each block is the same, say k, (ii) the block set can be partitioned into a family $\mathcal{R}$ of partial parallel classes, and (iii) each $R_i \in \mathcal{R}$ can be associated with a group $G_j \in \mathcal{G}$ so that R_i contains every point of $X \backslash G_j$ exactly once.*

If the blocks of a frame have a particular property then the frame is said to have that property and one typically introduces some notation to reflect that property. Thus Ge and Zhu [14] deal with the triplewhist frames (TWh frames), and Abel et al. [2] utilize (t, k) Generalized Whist Frames ((t, k)GWhFrames). It is the latter that is of interest here. These frames are such that every pair x and y from distinct groups appear together in exactly k blocks such that they appear exactly $t - 1$ times as teammates and exactly $k - t$ times as opponents.

2 Preliminaries

Suppose $v \equiv 1 \pmod{k}$. Then a (t, k)GWhD(v) is said to be Z-cyclic if the players are elements in Z_v and the rounds are cyclically generated so that round $j + 1$ is obtained by adding $+1 \pmod{v}$ to every element in round j. If $v \equiv 0 \pmod{k}$ a (t, k)GWhD(v) is said to be Z-cyclic if the players are elements in $Z_{v-1} \cup \{\infty\}$ and the rounds are cyclically generated via $+1 \pmod{v-1}$ with the convention $\infty + 1 = \infty$. Many authors use the term 1-rotational in this latter case. In both cases the entire design is completely determined by a single round which is typically termed the initial round. We adhere to the convention that the initial round of a Z-cyclic (t, k)GWhD($kn + 1$) is the round that omits 0 and the initial round of a Z-cyclic (t, k)GWhD(kn) is one of the rounds in which ∞ and 0 are partners.

Example 2.1 The initial round of a Z-cyclic $(2, 6)$GWhD(6) is given by a single game $(\infty, 0; 1, 4; 2, 3)$.

Definition 2.2 *A homogeneous $(v, k, 1)$-DM is a k by v array such that each row contains every element in Z_v exactly once and the set of differences of any two rows equals Z_v.*

Theorem 2.3 *If $gcd(v, k!) = 1$ then there exists a homogeneous $(v, k, 1)$-DM.*

Proof: Take as rows $Z_v, 2Z_v, 3Z_v, \ldots, \lceil \frac{k}{2} \rceil Z_v, -Z_v, -2Z_v, \ldots, -(k - \lceil \frac{k}{2} \rceil) Z_v$. ∎

As a convention we will always assume that a homogeneous $(v, k, 1)$-DM has a first column of zeros.

Definition 2.4 *Let $(a_{11}, \ldots, a_{1t}; \ldots; a_{e1}, \ldots, a_{et})$ denote any game in a (t, k)GWhD(v). If every such game is rewritten in the form $(a_{11}, a_{21}, \ldots, a_{e1}, a_{12}, a_{22}, \ldots, a_{e2}, \ldots, a_{1t}, a_{2t}, \ldots, a_{et})$ and if the design consisting of these rewritten blocks is a $(v, k, 1)$-resolvable perfect Mendelsohn design [16], $(v, k, 1)$-RPMD, then the (t, k) GWhD(v) is said to be directed.*

Definition 2.5 *Let $(b_1, b_2, \ldots, b_k)$ denote a block in a $(v, k, 1)$-RPMD. We consider the block to be cyclic in the sense of being a $(2k-1)$-tuple, $(s_1, s_2, \ldots, s_{2k-1})$, as follows : $s_i = b_i$, $1 \leq i \leq k$, and $s_{k+1} = b_i$, $1 \leq i \leq k-1$. Let $j \in \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k-1\}$ define the i-apart element of b_j to be s_m where $m > j$ and $m - j = i$.*

Theorem 2.6 *Let $v = kn+1$. If there exists a Z-cyclic (t, k)GWhD (v) that is directed then there exists a homogeneous $(v, k, 1)$-DM.*

Proof: Begin by rearranging the initial round games of the Z-cyclic (t, k)GWhD(v) to form the blocks of an initial resolution class of a $(v, k, 1)$-RPMD. Form a k by v array $M = (m_{ij})$ as follows : $m_{i1} = 0$, $i = 1, \ldots, k$; $m_{1j} = j - 1$, $j = 2, \ldots, v$; and $m_{ij} = m_{1j}$'s $(i-1)$-apart element in the initial resolution class, $i = 2, \ldots, k$, $j = 2, \ldots, v$. Label the rows of M as R_s, $s = 0, 1, \ldots, k-1$. It follows now that for $0 \leq u < w \leq k-1$, $R_w - R_u = \{(w-u)\text{-apart differences in the } (v, k, 1)\text{-RPMD}\} \cup \{0\}$. Thus by the definition of a cyclic $(v, k, 1)$-RPMD, $R_w - R_u = Z_v$. ∎

The following theorem is proven in [1]. The proof is constructive and the design is cyclic over the additive group in GF(v).

Theorem 2.7 *If $v = nk + 1$ is a power of a prime then there exists a directed (t, k) GWhD(v) for every t that divides k.*

Corollary 2.8 *If $v = nk + 1$ is a prime then there exists a Z-cyclic directed (t, k) GWhD(v) for every t that divides k.*

Corollary 2.9 *If $v = nk + 1$ is a prime then there exists a Z-cyclic directed (t, k) GWhD(v^n) for every t that divides k and for all $n \geq 1$.*

Proof: Apply induction on n utilizing Theorem 2.10, below. ∎

The next two theorems are generalizations of those found in [10].

Theorem 2.10 *Suppose there exists a Z-cyclic (t, k) GWhD $(ks_1 + 1)$, a Z-cyclic (t, k) GWhD$(ks_2 + 1)$ and a homogeneous $(ks_1+1, k, 1)$-DM. Then there exists a Z-cyclic (t, k) GWhD$((ks_1+1)(ks_2 + 1))$. If both of the input designs are directed then so is the final design.*

Proof: Let $A = (a_{ij})$ denote the DM and let IR_i denote the initial round games of the cyclic (t, k)GWhD$(ks_i + 1)$, $i = 1, 2$. For each game g in IR_1, construct the game $(ks_2 + 1)g$ (that is to say, multiply every player in g by $ks_2 + 1$ with the arithmetic taken modulo $((ks_1 + 1)(ks_2 + 1))$. For each game, $g = (g_1, g_2, \ldots, g_k)$, in IR_2 form the collection of games $(g_1+a_{1j}(ks_2+1), g_2 + a_{2j}(ks_2 + 1), \ldots, g_k + a_{kj}(ks_2 + 1))$, $j = 1, 2, \ldots, ks_1 + 1$. A simple contrapositive argument establishes that the union of the games so constructed is the initial round of a Z-cyclic (t, k)GWhD$((ks_1 + 1)(ks_2 + 1))$. Since the structure of the initial round games of both of the input designs is unaltered by this construction, it is clear that the final design is directed if the two input designs are. ∎

Theorem 2.11 *Suppose there exist a Z-cyclic (t, k) GWhD(ks_1), a Z-cyclic (t, k) GWhD$(ks_2{+}1)$ and a homogeneous $(ks_1{-}1, k, 1)$-DM. Then there exists a Z-cyclic (t, k) GWhD$((ks_1 - 1)(ks_2 + 1) + 1)$.*

Proof: Denote the difference matrix by (a_{ij}). Let IR_1 be the initial round of the Z-cyclic (t, k)GWhD(ks_1) and let IR_2 be the initial round of the Z-cyclic (t, k)GWhD$(ks_2 + 1)$. For each game, g, in IR_1, form the game $(ks_2 + 1)g$. Of course $(ks_2 + 1) * \infty = \infty$. For each game $g = (g_1, \ldots, g_k)$ in IR_2 form the games $(g_1 + a_{1j}(ks_2 + 1), \ldots, g_k + a_{kj}(ks_2 + 1))$, $j = 1, \ldots, ks_1 - 1$. As before, a straightforward contrapositive argument establishes that these games form the initial round of a Z-cyclic (t, k)GWhD$((ks_1 - 1)(ks_2 + 1) + 1)$. ∎

Definition 2.12 *Suppose $S = Z_v$, $v = hn$ and Z_v has a subgroup H of order h. Suppose a (t, k) GWhFrame(h^n) has a special partial resolution class (called the initial round) whose elements form a partition of $S \setminus H$ and such that all other partial resolution classes can be arranged in a cyclic order so that one can pass from one partial resolution class to the next by adding $+1$ (mod v) to each element. Such a (t, k) GWhFrame is said to be Z-cyclic and the partial resolution classes are called rounds.*

Theorems 2.10 and 2.11 allow one to construct many illustrations of Z-cyclic (t, k)GWhFrames.

Theorem 2.13 *Suppose $v_1 = ks_1 + 1$, $v_2 = ks_2 + 1$ are such that the hypotheses of Theorem 2.10 are satisfied. Then there exists a Z-cyclic (t, k) GWhFrame$(v_1^{v_2})$.*

Proof: From the initial round of the Z-cyclic (t, k)GWhD$((ks_1 + 1)(ks_2 + 1))$ remove all of the games constructed from IR_1. The initial round of a Z-cyclic (t, k)GWhFrame$(v_1^{v_2})$ with $H = \{0, v_2, 2v_2, \ldots, (v_1 - 1)v_2\}$ is formed from the remaining games. ∎

Theorem 2.14 *Suppose $v_1 = ks_1 - 1$, $v_2 = ks_2 + 1$ are such that the hypotheses of Theorem 2.11 are satisfied. Then there exists a Z-cyclic (t, k) GWhFrame$(v_1^{v_2})$.*

Proof: The construction is similar to that in the proof of Theorem 2.13. ∎

A particularly interesting special case of Theorem 2.14 is embodied in the following corollary.

Corollary 2.15 *Let $v = (k-1)u + 1$. If there exists a Z-cyclic (t, k) GWhD(v) for which the initial round contains a game $g = (g_1, \ldots, g_k)$ so that $\{g_1, \ldots, g_k\} = \{0, u, 2u, \ldots, (k-2)u\} \cup \{\infty\}$ then there exists a Z-cyclic (t, k) GWhFrame$((k-1)^u)$.*

Example 2.16 For each prime $p = 4n + 1$ there exists a Z-cyclic $(2, 4)$GWhFrame(3^p). This follows from Moore's Z-cyclic whist construction on $v = 3p + 1$ players [17] since his initial round always includes the game $(\infty, 0; p, 2p)$.

The next theorem employs the use of difference families. For information related to these designs, see [11].

Theorem 2.17 *If there exists a $(k(k-1)w+k-1, k-1, k, 1)$-DF over $Z_{k(k-1)w+k-1}$ and if there exists a Z-cyclic (t, k) GWhD(k) then there exists a Z-cyclic (t, k) GWhFrame$((k-1)^{kw+1})$.*

Proof: Using the DF and the Z-cyclic (t, k)GWhD(k) as input designs, apply the construction of Buratti-Zuanni (see Theorem 5.1 in [12]) to obtain the initial round of a Z-cyclic (t, k)GWhD $(k(k-1)w + k)$ that satisfies the conditions of Corollary 2.15. ∎

3 Frame Constructions for Z-Cyclic (t,k)GWhD(v)

The elegant methods of Ge and Zhu [14] easily carry over to the more general setting of (t,k)GWhD(v). The following is the generalization of their inflation theorem.

Theorem 3.1 *If there exists a Z-cyclic (t,k)GWhFrame(h^n) and if there exists a homogeneous $(q,k,1)$-DM then there exists a Z-cyclic (t,k)GWhFrame$((qh)^n)$.*

Proof: Let (a_{ij}) denote the $(q,k,1)$-DM. For each game $g = (g_1,\dots,g_k)$ in the initial round of the Z-cyclic (t,k)GWhFrame (h^n) construct the collection of games $(g_1+a_{1j}hn,\dots,g_k+a_{kj}hn)$, $j = 1,\dots,q$. ∎

The next theorem is, in some sense, a product theorem.

Theorem 3.2 *If there is a Z-cyclic (t,k)GWhFrame $(h^{\frac{v}{h}})$ and a Z-cyclic (t,k)GWhFrame$(u^{\frac{h}{u}})$. Then there exists a Z-cyclic (t,k)GWhFrame$(u^{\frac{v}{u}})$.*

Proof: Clearly for the hypothesis of the theorem to be satisfied it must be the case that $v = hm$ and $h = un$ where all symbols are positive integers. Let IR_1 denote the initial round of the Z-cyclic (t,k)GWhFrame$(h^{\frac{v}{h}})$ and IR_2 denote the initial round of the Z-cyclic (t,k)GWhFrame$(u^{\frac{h}{u}})$. Note that IR_1 is over Z_v and IR_2 is over Z_h. Replace each game g in IR_2 by mg and denote this new collection of games by IR_2^*. Since IR_2 misses the u multiples of n in Z_h, IR_2^* misses the u multiples of mn in Z_v. Thus it follows that $IR_1 \cup IR_2^*$ constitutes an initial round for a Z-cyclic (t,k)GWhFrame$(u^{\frac{v}{u}})$. ∎

The following two theorems show how to construct Z-cyclic (t,k)GWhD from Z-cyclic (t,k)GWhFrames.

Theorem 3.3 *Suppose there exists a Z-cyclic (t,k)GWhFrame(h^n) and a Z-cyclic (t,k)GWhD(h), $h \equiv 1 \pmod{k}$. Then there exists a Z-cyclic (t,k)GWhD(nh).*

Proof: For each game, g, in the initial round of the Z-cyclic (t,k)GWhD(h) form the game ng. Adjoin these games to the initial round of the Z-cyclic (t,k)GWhFrame(h^n) to get the initial round of the Z-cyclic (t,k)GWhD(nh). ∎

Theorem 3.4 *Suppose there is a Z-cyclic (t,k)GWhFrame(h^n) and a Z-cyclic (t,k)GWhD$(h+1)$, where $h \equiv k-1 \pmod{k}$. Then there exists a Z-cyclic (t,k)GWhD$(nh+1)$.*

Proof: For each game, g, in the initial round of the Z-cyclic (t,k)GWhD$(h+1)$ form the game ng (of course $n * \infty = \infty$) and adjoin these games to the initial round of the Z-cyclic (t,k)GWhFrame(h^n) to obtain the initial round of a Z-cyclic (t,k)GWhD$(nh+1)$. ∎

4 Some New Z-Cyclic $(2,6)$GWhD(v)

In this section we combine the materials of Sections 1, 2 and 3 to obtain several new Z-cyclic $(2,6)$GWhD(v). For reference we list some designs that appear in [3].

Example 4.1 The initial round of a Z-cyclic $(2,6)$GWhD(12) is given by the following two games:
$(\infty,0;8,10;1,5)$, $(7,2;3,4;9,6)$.

Example 4.2 The initial round of a Z-cyclic $(2,6)$GWhD(25) is given by the following four games:
$(18,3;9,13;14,16)$, $(23,24;12,15;6,17)$, $(22,5;19,1;2,7)$
$(11,20;21,8;4,10)$.

Example 4.3 The initial round of a Z-cyclic $(2,6)$GWhD(36) is given by the following six games:

$(\infty,0;16,25;34,2)$, $(27,28;3,33;14,20)$, $(23,6;22,12;8,29)$, $(7,26;30,10;15,17)$, $(9,21;13,24;11,18)$, $(5,1;4,31;19,32)$.

Example 4.4 The initial round of a Z-cyclic directed $(2,6)$GWhD(55) is given by the following nine games:

$(50,5;45,10;37,18)$, $(33,22;12,43;51,4)$, $(1,54;2,53;39,16)$, $(19,36;38,17;26,29)$, $(28,27;52,3;24,31)$, $(7,48;13,42;6,49)$, $(40,15;20,35;11,44)$, $(46,9;32,23;34,21)$, $(14,41;25,30;8,47)$.

Example 4.5 The initial round of a Z-cyclic $(2,6)$GWhD(66) is given by the following eleven games:

$(\infty,0;1,64;8,57)$, $(16,49;55,10;30,35)$, $(32,33;18,47;9,56)$, $(21,44;23,42;31,34)$, $(38,27;54,11;53,12)$, $(28,37;25,40;3,62)$, $(39,26;6,59;22,43)$, $(13,52;17,48;19,46)$, $(61,4;14,51;7,58)$, $(29,36;5,60;24,41)$, $(2,63;15,50;20,45)$.

Theorems 2.10 and 2.11 are quite powerful and have been used extensively to determine new Z-cyclic Wh(v) from known Z-cyclic designs. There is one case, however, in which these theorems generally do not apply. This situation occurs when $v=(k-1)u+1$, $u\equiv 1 \pmod k$. Oftentimes it is not known whether or not a homogeneous $((k-1)u,k,1)$-DM exists. Consequently, finding Z-cyclic (t,k)GWhD$((k-1)u+1)$ is usually a difficult task. As a case in point there are still many open questions about the existence of Z-cyclic $Wh(3u+1)$, $u=4n+1$, Moore's Construction, Example 2.16, notwithstanding.

Theorem 4.6 *For all $n\geq 1$ there exists a Z-cyclic directed $(2,6)$GWhD$((55)^n)$ and hence a homogeneous $((55)^n,6,1)$-DM.*

Proof: The proof is by induction on n. For $n = 1$ there is the Z-cyclic directed $(2,6)$GWhD(55) of Example 4.4. An application of Theorem 2.6 produces a homogeneous $(55,6,1)$-DM. Assume the theorem true for $n = s$. For $n = s+1$ apply Theorem 2.10 with $ks_1 + 1 = 55$ and $ks_2 + 1 = (55)^s$ to obtain a Z-cyclic directed $(2,6)$GWhD$((55)^{s+1})$. The homogeneous $((55)^{s+1}, 6, 1)$-DM follows from Theorem 2.6. ▮

Corollary 4.7 *For all* $n \geq 1$ *there exists a* Z*-cyclic* $(2,6)$*GWhD* $(25 \cdot (55)^n)$.

Proof: Let $n \geq 1$ be given. Apply Theorem 2.10 with $ks_1 + 1 = (55)^n$ and $ks_2 + 1 = 25$. ▮

The methodology presented in Sections 2 and 3 together with the Buratti-Zuanni Construction [12] and the difference families of Greig [15] enable one to construct many examples of Z-cyclic $(2,6)$GWhD$(5u+1)$.

Theorem 4.8 *Consider* t *to be such that* $4 \leq t \leq 416$ *and* $p = 12t+1$ *is a prime. Then for each* $q \in \{7, 13\}$ *there exists a* Z*-cyclic* $(2,6)$*GWhD*$(5qp+1)$.

Proof: If $p = 12t + 1$ is a prime, $4 \leq t \leq 416$, then there is a cyclic $6 - GDD(5^p)$ [15]. The points in this latter GDD (which is, in fact, a frame) are in Z_{5p}. The corresponding DF and the Z-cyclic $(2,6)$GWhD(6) of Example 2.1 provide the input for an application of Theorem 2.17. Thus we obtain a Z-cyclic $(2,6)$GWhFrame(5^p). For each $q \in \{7, 13\}$ there exists a homogeneous $(q,6,1)$-DM, so we can inflate this latter frame to obtain a Z-cyclic $(2,6)$GWhFrame$((5q)^p)$. Since there exist Z-cyclic $(2,6)$GWhD$(5q+1)$, $q \in \{7,13\}$, an application of Theorem 3.4 yields the Z-cyclic $(2,6)$GWHD$(5qp+1)$. ▮

We note that there are 157 primes of the form $12t+1$, $4 \leq t \leq 416$.

Theorem 4.9 *Let t and p be as in Theorem 4.8. Then there exists a Z-cyclic $(2,6)$ GWhD$((55)^n p)$, for all $n \geq 1$.*

Proof: As in the proof of Theorem 4.8 we can construct a Z-cyclic $(2,6)$GWhFrame(5^p). Inflate this latter frame by $11 \cdot (55)^{n-1}$ and use Theorem 3.3 along with Theorem 4.6 to produce the Z-cyclic $(2,6)$GWhD$((55)^n p)$. ∎

Now that we have a collection of Z-cyclic $(2,6)$GWhD(v) we can manipulate them to obtain others. Set $P = \{p = 12t + 1 :$ p is a prime, $4 \leq t \leq 416\} \cup \{55\}$.

Theorem 4.10 *Let v denote an arbitrary product of elements from $P \backslash \{55\}$. Then there exists a Z-cyclic $(2,6)$ GWhFrame(5^v).*

Proof: As in the proof of Theorem 4.8 there is a Z-cyclic $(2,6)$ GWhFrame(5^{p_1}) for all $p_1 \in P \backslash \{55\}$. Since there exists a homogeneous $(p_2, 6, 1)$-DM for all $p_2 \in P \backslash \{55\}$, one can inflate this latter frame to obtain a Z-cyclic $(2,6)$GWhFrame$((5p_2)^{p_1})$. Apply now Theorem 3.2 with $h = 5p_2$, $u = 5$, $v = 5p_1p_2$ to obtain a Z-cyclic $(2,6)$GWhFrame$((5)^{p_1p_2})$. The conclusion now follows by applying the latter result recursively. ∎

Theorem 4.11 *For all $n \geq 1$ there exists a Z-cyclic $(2,6)$ GWhD$(5(55)^n + 1)$.*

Proof: Let $n \geq 1$ be given. Then $5(55)^n + 1 = 11 \cdot 25 \cdot (55)^{n-1} + 1$. Apply Theorem 2.11 with $ks_1 = 12$ and $ks_2 + 1 = 25 \cdot (55)^{n-1}$. ∎

Theorem 4.12 *Let v be an arbitrary product of elements in P. Then there exists a Z-cyclic $(2,6)$ GWhD$(5v + 1)$.*

Proof: Define n by the requirement that $(55)^n | v$, $(55)^{n+1} \not| v$. Set $v_1 = v/(55)^n$. Begin with the Z-cyclic $(2,6)$GWhFrame(5^{v_1}) of Theorem 4.10. Inflate this frame by $(55)^n$. If $n > 0$, Theorem 4.11 combined with Theorem 3.4 gives the desired result. If $n = 0$, combine Example 2.1 with Theorem 3.4. ∎

Theorem 4.13 *Let $Q = 7^a 13^b$ where a, b are non-negative integers. Let v by an arbitrary product of elements in P and define n as in the proof of Theorem 4.12. Then there exists a Z-cyclic $(2,6)$GWhD$(5Qv+1)$ provided that if $n = 0$ then $a + b = 1$.*

Proof: Clearly it can be assumed that $a+b \geq 1$, for otherwise we have Theorem 4.12. Begin with a Z-cyclic $(2,6)$GWhFrame(5^{v_1}) of Theorem 4.12 where $v_1 = v/(55)^n$. Inflate this frame by $Q(55)^n$. If $n > 0$ then $5Q(55)^n = 11 \cdot 25 \cdot Q \cdot (55)^{n-1}$. An application of Theorem 2.10 with $ks_1 + 1 = 7^a$ if $a > 0$ or 13^b if $a = 0$ demonstrates the existence of a Z-cyclic $(2,6)$GWhD$(25 \cdot Q \cdot (55)^{n-1})$. Thus, via Theorem 2.11, there exists a Z-cyclic $(2,6)$GWhD$(11 \cdot 25 \cdot Q \cdot (55)^{n-1} + 1)$. Apply Theorem 3.4 to obtain the desired result. If $n = 0$ then Example 4.3 or Example 4.5, whichever is appropriate, combined with Theorem 3.4 yields the conclusion of the theorem. ∎

Acknowledgement The authors would like to acknowledge the contributions of the referees and thank them for their insights.

References

[1] R. J. R. Abel, N. J. Finizio and M. Greig, $(3,6)GWhD(v)$ - Existence Results, Discrete Math. (to appear).

[2] R. J. R. Abel, N. J. Finizio, M. Greig, and S. J. Lewis, Generalized Whist Tournament Designs, Discrete Math. (to appear).

[3] R. J. R. Abel, N. J. Finizio, M. Greig, and S. J. Lewis, $(2,6)GWhD(v)$ - Existence Results and Some Z-Cyclic Solutions, Congr. Numer. 144(2000), 5–39.

[4] R. J. R. Abel, N. J. Finizio, M. Greig, and S. J. Lewis, Pitch Tournament Designs and other BIBDs - Existence Results for the Case $v = 8n+1$, Congr. Numer. 138(1999), 175–192.

[5] R. J. R. Abel, N. J. Finizio, M. Greig, and S. J. Lewis, Pitch Tournament Designs and other BIBDs - Existence Results for the Case $v = 8n$, J. Combin. Des. 9(2001), 334–356.

[6] I. Anderson, "Combinatorial Designs and Tournaments", Oxford University Press, Oxford, 1997.

[7] I. Anderson, A hundred years of whist tournaments, J. Combin. Math. Combin. Comput. 19(1995), 129–150.

[8] I. Anderson, Whist Tournaments, in C.J. Colbourn and J.H. Dinitz (eds.), *CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton FL, 1996, 504–508.

[9] I. Anderson, Some Cyclic and 1-Rotational Designs, preprint.

[10] I. Anderson, N.J. Finizio, and P. A. Leonard, New Product Theorems for Z-Cyclic Whist Tournaments, J. Combin. Theory, Ser. A 88(1999), 162–166.

[11] T. Beth, D. Jungnickel, and H. Lenz, "Design Theory", Cambridge University Press, 1993.

[12] M. Buratti and F. Zuanni, Perfect Cayley Designs as Generalizations of Perfect Mendelsohn Designs, Des. Codes. Cryptogr. 23(2001), 233–247.

[13] S. Furino, Y. Miao, and J. Yin, "Frames and Resolvable Designs : Uses, Constructions, and Existence", CRC Press, Boca Raton, FL, 1996.

[14] G. Ge and L. Zhu, Frame constructions for Z-cyclic triple-whist tournaments, Bull. ICA 32(2001), 53–62.

[15] M. Greig, Some group divisible design constructions, J. Combin. Math. Combin. Comput. 27(1998), 33–52.

[16] E. Mendelsohn, Mendelsohn Designs, in C.J. Colbourn and J.H. Dinitz (eds.), *CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, FL, 1996, 388–393.

[17] E. H. Moore, Tactical Memoranda I-III, Amer. J. Math. 18(1896), 264–303.