§ 4.5 Moment - Generating Functions

A shortcut to calculating the moments of a distribution.

<u>Defn</u> The <u>moment-generating function</u> (MGF) $M_X(t)$ of a random variable X is given by

$$M_X(t) = E(e^{tX})$$

When X is discrete, we have (by Thm 4.1)

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} \cdot f(x).$$

When X is cts, we have (by Thm 4.1 again)

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx.$$

As an <u>object</u>, $M_X(t)$ is a function from $\mathbb{R} \to \mathbb{R}$ given by the rule

$$t \longmapsto E(e^{tX}).$$

Why the name moment-generating $f^{\underline{n}}$?

Recall that e^{tx} has McLaurin series expansion

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^r}{r!} + \cdots$$

In the discrete case, we get

$$M_X(t) = E(e^{tX})$$

$$= \sum_x e^{tx} \cdot f(x).$$

$$= \sum_x \left(1 + tx + \frac{t^2x^2}{2!} + \cdots + \frac{t^r}{r!}\right) \cdot f(x)$$

$$= \sum_x f(x) + t\sum_x x f(x) + \frac{t^2}{2!}\sum_x x^2 f(x) + \cdots$$

$$\cdots + \frac{t^r}{r!}\sum_x x^r f(x) + \cdots$$

(formal interchange of sums ok)

$$= 1 + \mu t + \mu_2' \frac{t^2}{2!} + \cdots + \mu_r' \frac{t^r}{r!} + \cdots$$

Thus the fn $M_X(t)$ itself has a McLaurin series about 0 and the coefficient of t^r in this series is $\frac{\mu_r'}{r!}$ (where μ_r' is the r-th moment of X about 0).

A similar argument holds in the cts case.

Recall that if ~~a fn~~ $g(t)$ has McLaurin series

$$g(t) = \sum_{r=0}^{\infty} a_r \frac{t^r}{r!}$$

about 0, then

$$a_r = \left. \frac{d^r g(t)}{dt^r} \right|_{t=0}.$$

Applying this to the MGF of X gives,

<u>Theorem 4.9</u>

$$\left. \frac{d^r M_X(t)}{dt^r} \right|_{t=0} = \mu_r'$$

This result shows us how to obtain the moments of X from the MGF.

<u>Ex</u> Find the MGF of the r.v. whose pdf is

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

and use it to find an expression for μ_r'.

<u>Soln.</u> By defn

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_0^{\infty} e^{-x(1-t)}$$

$$= \frac{1}{1-t} \quad \text{for } t<1 \qquad \left(\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a} \text{ for } a>0 \right)$$

It is well known that

$\frac{1}{1-t}$ has McLaurin series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots + t^r + \cdots, \qquad |t| < 1$$

Thus

$$M_X(t) = 1 + t + t^2 + \cdots + t^r + \cdots$$

$$= 1 + 1!\frac{t}{1!} + 2!\frac{t^2}{2!} + \cdots + r!\frac{t^r}{r!} + \cdots$$

and so $\mu_r' = r!$, $r = 0, 1, 2, \ldots$

Often the main difficulty with using MGFs is not finding $M_X(t)$, but finding its McLaurin series. However, if we only require a few of the moments, this problem can be got round by appealing to Thm 4.9 and differentiating.

Ex. Suppose X is discrete with pdf

$$f(x) = \frac{1}{8}\binom{3}{x}, \qquad x = 0, 1, 2, 3.$$

Find the MGF of X and use it to determine μ_1' and μ_2'.

Soln. By defn.

$$M_X(t) = E(e^{tX}) = \frac{1}{8}\sum_{x=0}^{3} e^{tx}\binom{3}{x}$$

$$= \frac{1}{8}(1 + 3e^t + 3e^{2t} + e^{3t})$$

$$= \frac{1}{8}(1 + e^t)^3$$

It would be rather hard to find the McLaurin series of this fn. However, by Thm 4.9

$$\mu_1' = M_X'(0) = \frac{3}{8}(1+e^t)^2 \cdot e^t \Big|_{t=0} = \frac{3}{2}$$

and

$$\mu_2' = M_X''(0) = \frac{3}{4}(1+e^t)\cdot e^{2t} + \frac{3}{8}(1+e^t)^2\cdot e^t \Big|_{t=0}$$

$$= \frac{3}{4}\cdot 2 + \frac{3}{8}\cdot 4$$

$$= 3.$$

Finally, a useful result for calculating MGFs

<u>Thm 4.10</u> If a and b are constants, then

1. $M_{X+a}(t) = E\left(e^{(X+a)t}\right) = e^{at} M_X(t)$

2. $M_{bX}(t) = E\left(e^{bXt}\right) = M_X(bt)$

3. $M_{\frac{X+a}{b}}(t) = E\left(e^{\left(\frac{X+a}{b}\right)t}\right) = e^{\frac{a}{b}t} M_X\left(\frac{t}{b}\right)$

Pf. Exercise. (easy).

Part 1 of this thm is useful when $a = -\mu$ and part 2 is useful when $a = -\mu$, $b = \sigma$ in which case

$$M_{\frac{X-\mu}{\sigma}}(t) = e^{-\frac{\mu b}{\sigma}} M_X\left(\frac{t}{\sigma}\right)$$

This trick is useful for using MGFs to 'prove' the central limit theorem (see later).

As an exercise, show that if X has mean μ and variance σ, then

$$\frac{X-\mu}{\sigma}$$

has mean 0 and variance 1.