§ 4.4 Markov's and Chebyshev's Inequalities

Thm 4.7.5 (Markov's Inequality)

If X is a r.v. with mean μ and pdf $f(x)$ with $f(x) = 0$ for $x < 0$, then for any positive constant a,

$$P(X \geq a) \leq \frac{\mu}{a}$$

Pf. (Cts case - discrete is similar).

$$P(X \geq a) = \int_a^\infty f(x)\,dx$$

$$\leq \int_a^\infty \frac{x}{a} f(x)\,dx \qquad \text{as } \frac{x}{a} f(x) \geq f(x) \text{ when } x \geq a!$$

$$= \frac{1}{a} \int_a^\infty x f(x)$$

$$\leq \frac{1}{a} \int_0^\infty x f(x)$$

$$= \frac{1}{a} \int_{-\infty}^\infty x f(x) \quad \text{as} \quad f(x) = 0 \quad \text{when } x < 0$$

$$= \frac{1}{a} E(X)$$

$$= \frac{1}{a} \mu . \qquad \blacksquare$$

Ex. For the r.v. X we had earlier with pdf

$$f(x) = \begin{cases} \frac{4}{\pi(1+x^2)}, & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases},$$

use Markov to estimate $P(X \geq \frac{1}{2})$.

Sol. Had $\mu = \frac{\ln 4}{\pi} \approx .4413$ from earlier

By Markov $P(X \geq \frac{1}{2}) \leq \frac{\mu}{\frac{1}{2}} \approx .8826$

This illustrates the first main use of Markov's Inequality, namely that the estimates from Markov are often easier to calculate than the actual probabilities $P(X \geq a)$, especially if we already know μ.

The second main use of Markov is as an aid in proving Chebyshev's Inequality, which shows us how σ or σ^2 are indicative of the spread or dispersion of a r.v.

<u>Theorem 4.8</u> (Chebyshev's Inequality)

If μ and σ are the mean and std. deviation (resp.) of the r.v. X, then for any positive constant k

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}, \quad \sigma \neq 0$$

or

$$P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}, \quad \sigma \neq 0.$$

Pf Note that the second statement follows immediately from the first as we are dealing with complementary events.

To show the first, let Y be the r.v. $Y = (X-\mu)^2$.

Then since Y can only take on values ≥ 0, if we let g denote the pdf of y, then clearly $g(x) = 0$ for $x < 0$ (or, to be more exact, we can require that this be so).

Also, if we let ν be the mean of Y, then

$$\nu = E(Y) = E((X-\mu))^2$$
$$= \sigma^2 \quad \text{(by defn of } \sigma^2\text{)}.$$

We now apply Markov to Y with $a = k^2\sigma^2$ to get.

$$P(|X-\mu| \geq k\sigma)$$

$$= P((X-\mu)^2 \geq k^2\sigma^2) \qquad \text{(same event!)}$$

$$= P(Y \geq k^2\sigma^2)$$

$$\leq \frac{\nu}{k^2\sigma^2}$$

$$= \frac{\sigma^2}{k^2\sigma^2}$$

$$= \frac{1}{k^2} \quad \text{as } \sigma \neq 0. \qquad \blacksquare$$

One way of stating the second part of Chebyshev is that for any positive const k, the prob is <u>at least</u> $1-\frac{1}{k^2}$ that X will take on a value within k standard deviations of the mean.

Ex. If X is a cts. r.v. with pdf

$$f(x) = \begin{cases} 630x^4(1-x)^4, & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

find the prob. that it will take on a value within 2 std deviations of the mean and compare this with the lower bound given by Chebyshev.

Soln. Integration shows that $\mu = \frac{1}{2}$

and $\sigma^2 = \frac{1}{44}$ so $\sigma = \sqrt{\frac{1}{44}} \approx .15$.

Hence the prob. that X will take on a value within 2 std. deviations of the mean is (approx)

$$P([\tfrac{1}{2} - 2\times .15, \tfrac{1}{2} + 2\times .15])$$

$$= P([.2, .8]) = \int_{.2}^{.8} 630x^4(1-x)^4\,dx \approx .96$$

On the other hand, Chebyshev gives

$$P\left(|X - .5| > 2 \times .15\right) \geq 1 - \frac{1}{2^2}$$

$$= .75.$$