§ 3.4 Probability Density Functions

The defn of probability density fns for continuous r.v.'s is similar to that for discrete r.v.'s, except for a couple of 'surprises'!

<u>Defn 3.4</u> A fn $f: \mathbb{R} \longrightarrow \mathbb{R}$ is called a <u>probability density function</u> of the continuous random variable X iff

$$P([a,b]) = \int_a^b f(x)\,dx$$

Often called <u>densities</u> or just PDF(s).

Note that $f(c)$ does <u>not</u> give $P(X=c)$ as it does in the discrete case.

For cts r.v.'s we always have

$$P(X=c)=0 \quad \text{for any } c \in \mathbb{R}.$$

(e.g. the highway example).

Because of this, we can change 'some' of the values of $f(x)$ and still have a PDF for X. Hence, the PDF of X is <u>not</u> uniquely defined (as it is in the discrete case).

Similarly, it doesn't matter from the point of view of probability whether or not we include the endpts of an interval.

<u>Thm 3.4</u> If X is a cts r.v. and $a, b \in \mathbb{R}$, then

$$P([a,b]) = P([a,b)) = P((a,b]) = P((a,b)).$$

We also have an analogue of Thm 3.1 from the last section.

<u>Thm 3.5</u> A fn f can serve as the PDF of a cts r.v. X if its values, $f(x)$, satisfy

1. $f(x) \geq 0, \quad -\infty < x < \infty$,
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Ex. If X has the density

$$f(x) = \begin{cases} ke^{-3x}, & x > 0 \\ 0, & x \le 0 \end{cases},$$

find k and $P([0.5, 1])$.

Soln. By Thm 3.5, must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\text{Now } \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} k e^{-3x}\,dx$$

$$= \lim_{t \to \infty} \int_{0}^{t} k e^{-3x}\,dx$$

$$= k \lim_{t \to \infty} \int_{0}^{t} e^{-3x}\,dx$$

$$= k \lim_{t \to \infty} \left[\frac{e^{-3x}}{-3} \right]_{0}^{t}$$

$$= k \lim_{t\to\infty} \left[\frac{e^{-3t}}{-3} - \left(\frac{-e^{0}}{3}\right) \right]$$

$$= k \lim_{t\to\infty} \left(\frac{1}{3} - \frac{e^{-3t}}{3} \right)$$

$$= \frac{k}{3}.$$

Hence in order to have $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we need

$$\frac{k}{3} = 1$$

ie $\quad k = 3.$

We then get

$$P([0.5, 1]) = \int_{0.5}^{1} 3e^{-3x}\,dx = \left[-e^{-3x}\right]_{0.5}^{1}$$

$$= -e^{-3} + e^{-0.5} \approx 0.173.$$

Actually, the r.v. in this example cannot assume negative values, but we extended its PDF to all of $\mathbb{R}$.

As in the discrete case, we are often interested in $P(X \leq x)$.

<u>Defn 3.5</u> If X is a cts r.v. and the value of a PDF for X at t is $f(t)$, then the fn given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < t < \infty$$

is called the <u>cumulative distribution function</u> (CDF) of X.

Note that although $f(t)$ is not uniquely defined, $F(x)$ is as $P(X \leq x)$ is uniquely defined!

As with the discrete case we can say that

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1 \quad \text{and}$$

$$a, b \in \mathbb{R}, \quad a < b \Rightarrow F(a) \leq F(b)$$

(Almost) Formally, a continuous random variable is one whose <u>CDF is continuous</u>.

Also have

Thm 3.6 If $f(x)$, $F(x)$ are a PDF and the CDF for a cts r.v. X, then

$$P([a,b]) = F(b) - F(a), \quad a,b \in \mathbb{R},\ a \leq b$$

and

$$f(x) = \frac{dF(x)}{dx}$$

where this derivative exists.

Pf. (Not fully rigorous)

~~For~~ first part

$$(-\infty, b] = (-\infty, a) \cup [a, b] \quad \text{(disjoint)}$$

So

$$P((-\infty,b]) = P((-\infty,a)) + P([a,b])$$
$$= P((-\infty,a]) + P([a,b]) \quad \text{by Thm 3.4}$$

ie

$$F(b) = F(a) + P([a,b]), \text{ so}$$

$$P([a,b]) = F(b) - F(a) \text{ as desired.}$$

For second part, assume f is cts. at x.

$\Rightarrow$ if t is near x, then $f(t) \approx f(x)$.

Now suppose h is small $(\neq 0)$ (+ve or −ve).

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_{-\infty}^{x+h} f(t)\,dt - \int_{-\infty}^{x} f(t)\,dt \right)$$

$$= \frac{1}{h} \int_{x}^{x+h} f(t)\,dt$$

$$\approx \frac{1}{h} \cdot h\, f(x)$$

$$= f(x).$$

Hence if f is cts. at x,

$$\lim_{h \to 0} \frac{F(x+h)-F(x)}{h}$$ exists (ie F is differentiable at x)

and $$\frac{dF(x)}{dx} = f(x). \qquad \blacksquare$$

Ex. Find the CDF of the r.v. X of the last example and use it to reevaluate $P([0.5,1])$.

Soln. For $x>0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x 3e^{-3t}\,dt = \left[-e^{-3t}\right]_0^x = 1-e^{-3x}$$

and since $F(x)=0$ for $x \le 0$, we can write

$$F(x) = \begin{cases} 0, & x \le 0 \\ 1-e^{-3x}, & x>0 \end{cases}$$

By Thm 3.9, $P([0.5,1]) = F(1)-F(0.5)$

$$= (1-e^{-3}) - (1-e^{-1.5})$$

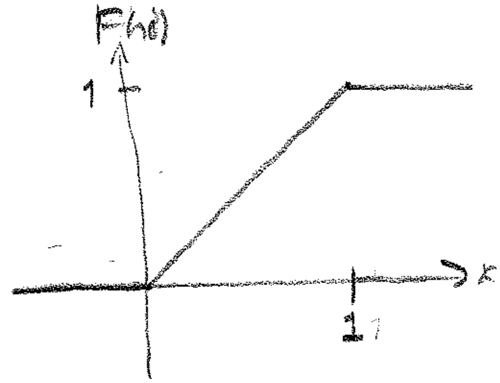
$$= e^{-1.5} - e^{-3}$$

$$= 0.173$$

(same answer as before). $\square$

Ex. Find a PDF for the r.v. whose CDF is

$$F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 < x < 1 \\ 1, & x \ge 1 \end{cases}$$



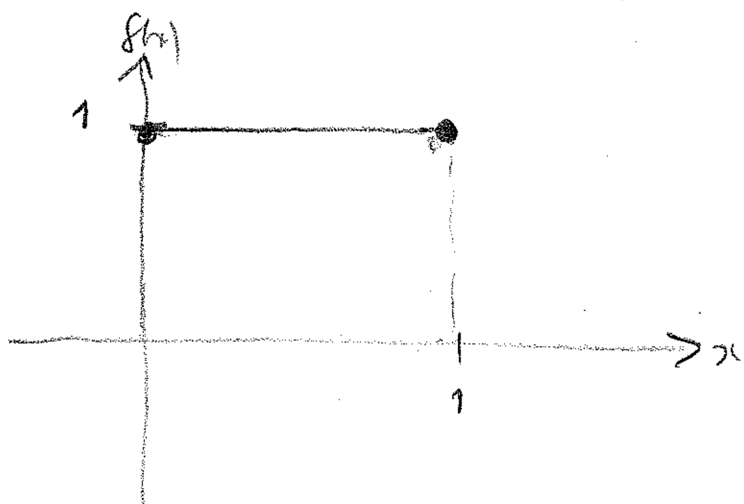
Soln. F is diff everywhere except at 0,1.

By Thm 3.9, we have

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

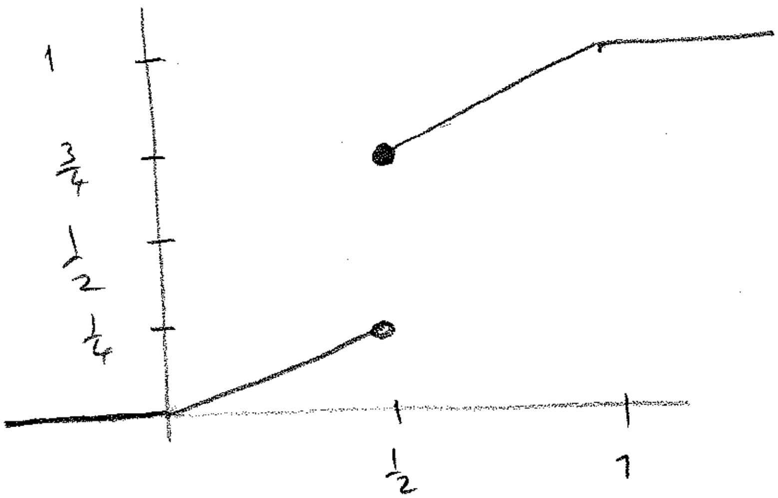
Still need to define f at 0,1. However, as we saw earlier, the particular values of f at 0,1 will not affect the probs. of events.

So set $f(0) = f(1) = 1$.



Also have r.v.'s which are neither discrete nor cts.

e.g. Consider a r.v. with CDF



This r.v. takes on uncountably many values so it is not discrete.

However, $P(\{\frac{1}{2}\}) = \frac{1}{2} > 0$, so it is not cts either (prob. of $\{\frac{1}{2}\}$ is the height of the jump in the CDF at $\frac{1}{2}$).

In our course, we shall limit ourselves to r.v.s which are discrete or cts with the CDF's of the cts. r.v.'s being piecewise differentiable (ie differentiable on all of $\mathbb{R}$ except (possibly) at finitely many values).