Note that convex fns need not be diff e.g $|x|$.

However, $|x|$ is 'nearly' diff in the sense that it only fails to be diff at 1 pt.

Makes sense to ask if the set of pts. where a general convex fn fails to be diff is small.

4.29 <u>Def</u>ⁿ f defined on an int I, $c \in I$. The <u>left</u> & <u>right derivatives</u> of f at c are denoted & defined by

$$f'_-(c) = \lim_{x \to c_-} \frac{f(x) - f(c)}{x - c}, \quad f'_+(c) = \lim_{x \to c_+} \frac{f(x) - f(c)}{x - c}.$$

Obviously f is diff at c iff $f'_-(c)$, $f'_+(c)$ both exist and are equal.

4.30 Thm If f is convex on (a,b) open, then f has one-sided derivatives at each pt of (a,b), $f'_-(x) \le f'_+(x)$ $\forall x \in (a,b)$ and f'_-, f'_+ are incr on (a,b).

Pf. Fix $c \in (a,b)$. Define φ on $(a,b)\setminus\{c\}$ by

$$\varphi(x) = \frac{f(x)-f(c)}{x-c}.$$

Let $A = \{\varphi(x) ; a < x < c\}$ $\quad (= \varphi((a,c)))$

$B = \{\varphi(x) ; c < x < b\}$ $\quad (= \varphi((c,b)))$.

By Lemma 4.27, φ is incr on (a,c) and also on (c,b).

Also each element of A is a l.b for B and each element of B is an u.b. for A.

Combining this with Th 3.31

$$f'_-(c) = \varphi(c-) = \sup A \le \inf B = \varphi(c+) = f'(c).$$

Shows f has 1-sided derivs. at c and $f'_-(c) \le f'_+(c)$.

Now supp. $x_1, x_2 \in (a,b)$ with $x_1 < x_2$.
Let v be any pt between x_1 & x_2.
By Lemma 4.27 & results from first part of pf,

$$f'_-(x_1) \le f'_+(x_1) \le \frac{f(v)-f(x_1)}{v-x_1} \le \frac{f(v)-f(x_2)}{v-x_2}$$
$$\le f'_-(x_2) \le f'_+(x_2).$$

Shows both f'_- & f'_+ are incr. on (a,b). $\boxed{/\!/}$

<u>4.31 Thm</u> If f is convex on (a,b) open, then the set of pts where f is not diff is ctable.

Pf. Since f'_+ is incr on (a,b), from prev. result, by Th 3.33, the set of discty s of f'_+ is countable.

To finish pf, suffices to show f is diff at each pt where f'_+ is cts.

So let c be a pt where f'_+ is cts and let $\varepsilon > 0$. Since f'_+ is cts at c $\exists\, \delta > 0$ s.t. $|f'_+(x) - f'_+(c)| < \varepsilon$ $\forall x \in (c-\delta, c+\delta)\ (\cap\ (a,b))$.

Supp that $c - \delta < x < c$.

From pf of prev. result (last ineq.), $f'_+(x) \leq f'_-(c)$. Thus

$$f'_+(c) - \varepsilon < f'_+(x) \leq f'_-(c) \leq f'_+(c).$$

Since $\varepsilon > 0$ was arb, we find that $f'_-(c) = f'_+(c)$ which shows f is diff at c. $\blacksquare$