Mth 435 Homework 5 - Metric Spaces

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Due Monday December 3, 2012

All problems are worth 20 points, including bonus problems which are extra credit.

An example of a complete metric space.

Let $\mathcal{C}[a, b]$ denote the space of all continuous real-valued functions on the closed interval $[a, b]$ and for $f, g \in \mathcal{C}[a, b]$ define the distance between f and g by

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

1. Show that this way of measuring distance between functions makes $\mathcal{C}[a, b]$ into a metric space.

2. Show that this metric space is complete.

   Hints:

   Take a Cauchy sequence $\{f_n\}$ with respect to this metric. Show that for each fixed $x \in [a, b]$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ and deduce something from a fact you know well to get a candidate limit function.

   Use a 3ϵ trick to deduce that this candidate limit function is continuous.

   The last thing you need to do is show that this sequence of functions converges to the limit function in the required metric (and not just pointwise which is what the first hint tells you). This can be done by letting n or m in the Cauchy criterion for the sequence $\{f_n(x)\}$ tend to infinity and observing that the rate of convergence will not depend on x.