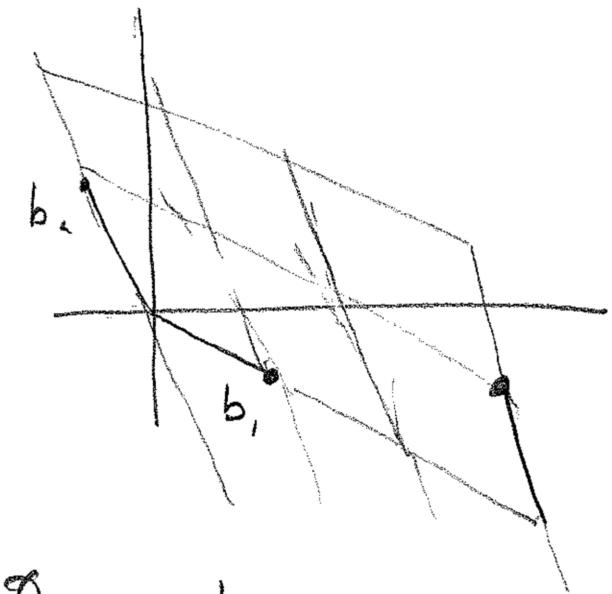
4.7 Change of Basis

If $\mathcal{B}$ is a basis for an n-dim vector space (v.s.) V, then each x in V is identified uniquely by its $\mathcal{B}$-coordinate vector $[x]_{\mathcal{B}}$ (in $\mathbb{R}^n$)

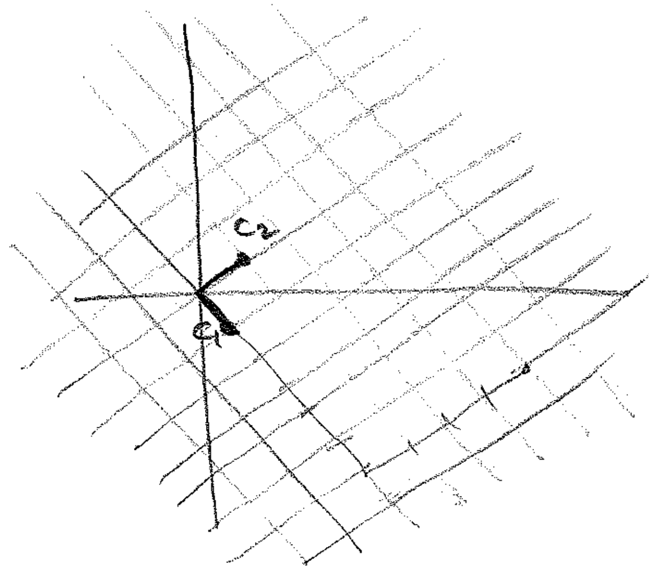
Sometimes a problem is formulated in terms of one basis $\mathcal{B}$ but a solution is required in terms of another basis $\mathcal{C}$. Hence we need to know how the coordinate representation $[x]_{\mathcal{B}}$ and $[x]_{\mathcal{C}}$ are related.

Example 0



$\mathcal{B}$ coords

$x = 3b_1 + b_2$



$\mathcal{C}$ coords

$x = 6c_1 + 4c_2$

By linearity our problem is solved if we can find b_1, b_2 in terms of c_1, c_2.

<u>Example 1</u> Consider two bases

$\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ for

a (2-dim) vector space V s.t.

$$b_1 = 4c_1 + c_2 \quad , \quad b_2 = -6c_1 + c_2 . \qquad (1)$$

Suppose

$$x = 3b_1 + b_2 \qquad (2)$$

ie suppose

$$[x]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} .$$

Find $[x]_{\mathcal{C}}$.

Soln. Apply the coord. mapping determined by $\mathcal{C}$ to x

$$[x]_{\mathcal{C}} = [3b_1 + b_2]_{\mathcal{C}}$$

$$= 3[b_1]_{\mathcal{C}} + [b_2]_{\mathcal{C}} \quad \text{by linearity.}$$

Write as a matrix eq^{n} with b_1, b_2 as the columns of the matrix

$$[x]_{\mathcal{C}} = \left[[b_1]_{\mathcal{C}} \; [b_2]_{\mathcal{C}} \right] \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \left[[b_1]_{\mathcal{C}} \; [b_2]_{\mathcal{C}} \right] [x]_{\mathcal{B}} \quad \text{by } (2)$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{by } (1)$$

$$= \begin{bmatrix} 6 \\ 4 \end{bmatrix} \quad \text{(same as in pictures).}$$

<u>Theorem 15</u> Let $\mathcal{B} = \{b_1, \_\_, b_n\}$ and $\mathcal{C} = \{c_1, \_\_\_, c_n\}$ be bases for an n-dim v.s. V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ such that

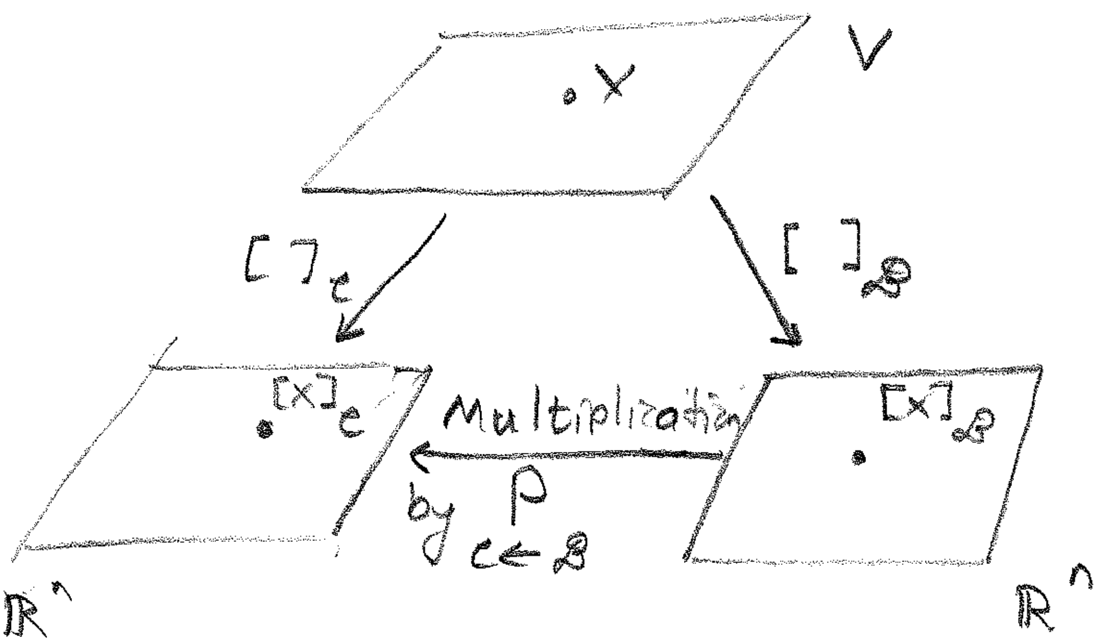
$$[x]_{\mathcal{C}} = \underset{\mathcal{C} \leftarrow \mathcal{B}}{P} [x]_{\mathcal{B}},$$

The columns of $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ are the C-coord. vectors of the vectors in the basis $\mathcal{B}$.

i.e.

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left[[b_1]_{\mathcal{C}} \; [b_2]_{\mathcal{C}} \; \cdots\cdots \; [b_n]_{\mathcal{C}} \right]$$

The matrix $\underset{C \leftarrow \mathcal{B}}{P}$ is called the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. Multiplication by $\underset{C \leftarrow \mathcal{B}}{P}$ changes $\mathcal{B}$-coords to $\mathcal{C}$-coords.



Then, by defn.

$$x_1 c_1 + x_2 c_2 = b_1$$

$$\Rightarrow \quad [c_1 \;\; c_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

and similarly

$$[c_1 \;\; c_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b_2$$

Thus

$$[c_1 \;\; c_2]\overbrace{\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}}^{\underset{C\leftarrow B}{P} \,=} = [b_1 \;\; b_2]$$

or

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = [c_1 \;\; c_2]^{-1}[b_1 \;\; b_2]$$

$$\underset{C\leftarrow B}{P} = \begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1}\begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

The columns of $\underset{C \leftarrow \mathcal{B}}{P}$ are lin. ind. as they are the coord. vectors of the lin. ind. ~~set~~ $\mathcal{B}$. Since $\underset{C \leftarrow \mathcal{B}}{P}$ is square, it must be invertible by the Invertible Matrix Theorem.

Left multiplying by $\left(\underset{C \leftarrow \mathcal{B}}{P}\right)^{-1}$ gives

$$\left(\underset{C \leftarrow \mathcal{B}}{P}\right)^{-1}\left(\underset{C \leftarrow \mathcal{B}}{P}\right)[X]_{\mathcal{B}} = \left(\underset{C \leftarrow \mathcal{B}}{P}\right)^{-1}[X]_C$$

$$\Leftrightarrow \quad [X]_{\mathcal{B}} = \left(P_{C \leftarrow \mathcal{B}}\right)^{-1}[X]_C$$

or

$$\left(\underset{C \leftarrow \mathcal{B}}{P}\right)^{-1}[X_C] = [X]_{\mathcal{B}}.$$

Thus $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$ changes $\mathcal{C}$-coords. into $\mathcal{B}$-coords and so

$$\boxed{\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}}$$

An Important Special Case

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for V & $\mathcal{E} = \{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$, then

$$[b_i]_{\mathcal{E}} = b_i \quad , \quad 1 \leq i \leq n$$

and so

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \Big[[b_1]_{\mathcal{E}} \cdots [b_n]_{\mathcal{E}} \Big] = \big[b_1 \cdots b_n \big] = P_{\mathcal{B}}$$

where $P_{\mathcal{B}}$ is the change of coords matrix of section 4.4.

In particular this works if $V = \mathbb{R}^n$ & $\mathcal{B}$ is a basis for $\mathbb{R}^n$.

Example 2.

Let $b_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $c_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{b_1, b_2\}$, $\mathcal{C} = \{c_1, c_2\}$.

Find the change of co-ordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Soln. The matrix $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ involves the $\mathcal{C}$-coord. vectors of b_1 and b_2.

So let

$$[b_1]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad [b_2]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This can be done either by using the formula for the inverse of a 2×2 matrix (which is easy but only works for $\mathbb{R}^2$) or as in the book by row operations (harder but more general)

Form the augmented matrix

$$[c_1 \; c_2 \mid b_1 \; b_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

& carry out row operations until the left half is I_2. Get

$$\left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

[As in section 2.2, the row operations correspond to multiplying on the left by a product of elementary matrices which must be $[c_1 \; c_2]^{-1}$]

Check

$$\begin{bmatrix} 1 & 3 \\ -4 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{-5-(-12)} \begin{bmatrix} -5 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -9 & -5 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 42 & 28 \\ -35 & -21 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix} \checkmark$$

Hence $\quad [b_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad [b_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

and

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

Moral to find $\underset{C \leftarrow \mathcal{B}}{P}$, form the augmented matrix $[c_1\ c_2 \mid b_1\ b_2]$ & row reduce the first half to I_2 so that the second half is automatically $\underset{C \leftarrow \mathcal{B}}{P}$.

ie

$$[c_1\ c_2 \mid b_1\ b_2] \sim [I_2 \mid \underset{C \leftarrow \mathcal{B}}{P}].$$

An analagous procedure works for $\mathbb{R}^n$ where we have bases with n vectors.

Example 3

Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$

and let $\mathcal{B} = \{b_1, b_2\}$, $\mathcal{C} = \{c_1, c_2\}$ be two bases for $\mathbb{R}^2$.

a. Find the change of coord. matrix from $\mathcal{C}$ to $\mathcal{B}$

b. Find the change of coord matrix from $\mathcal{B}$ to $\mathcal{C}$.

Sol^n: a. Since we need $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$ rather than $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$, we compute

$$[b_1\ b_2 \mid c_1\ c_2] = \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

and so $\underset{\mathcal{B} \leftarrow \mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$.

b. From earlier

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

$\square$

Note that for bases $\mathcal{B}$, $\mathcal{C}$ of $\mathbb{R}^n$ & $x \in \mathbb{R}^n$, from section 4.4 we have

$$P_{\mathcal{B}}\,[x]_{\mathcal{B}} = x, \quad P_{\mathcal{C}}\,[x]_{\mathcal{C}} = x \quad \text{and}$$

$$[x]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} x$$

Thus $\quad [x]_{\mathcal{C}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}\,[x]_{\mathcal{B}}$

and since the change of basis matrix is unique (Thm 15).

$$\boxed{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}.}$$

Can use this to find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ but the work is the same as before.