1.6 Linear Independence

Recall, for example, that the solution sets of

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are identical.

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only one*?

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

↑

linear dependence relation

(when weights are all not zero)

EXAMPLE 1 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: a. Examine solution of

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable, there are nontrivial solutions. So $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent (and not linearly independent).

(b) Reduced echelon form: $\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = \\ x_2 = \\ x_3 \end{array}$

Choose x_3 to be any nonzero number. Say $x_3 =$ ___. Then $x_1 =$ ____ and $x_2 =$ ____.

Therefore

$$\_\_\_\_ \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

Linear Independence of Matrix Columns

In last example, note that the linear dependence relation

$$-33\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be written as the matrix equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}\begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

> The columns of matrix A are linearly independent if and only if the equation of $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

EXAMPLE 2 Determine if the columns of $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 2 & 9 & 5 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$ are linearly dependent.

Solution: Corresponding augmented matrix of $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 0 \\ 0 & 2 & 9 & 5 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

Answer:

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. *A Set of One Vector*

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = 0$ is the ___________________________.

So $\{\mathbf{v}_1\}$ is linearly independent when $\mathbf{v}_1 \neq \mathbf{0}$.

2. *A Set of Two Vectors*

EXAMPLE 3 Let $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.
b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 =$ ___$\mathbf{u}_1$. Therefore

$$\_\_\_\mathbf{u}_1 + \_\_\_\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly ____________________ set.

(b) Suppose

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

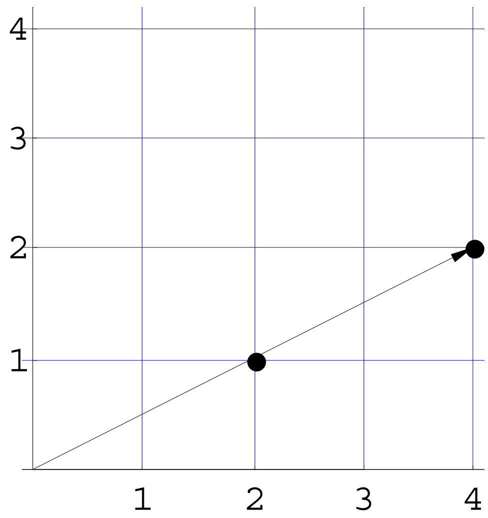
Then $\mathbf{v}_1 = \frac{\quad}{\quad}\mathbf{v}_2$ if $c_1 \neq 0$. But this is impossible since $\mathbf{v}_1$ is not a multiple of $\mathbf{v}_2$ which means $c_1 =$ ____.

Similarly, $\mathbf{v}_2 = \frac{\quad}{\quad}\mathbf{v}_1$ if $c_2 \neq 0$. But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $c_2 = 0$.

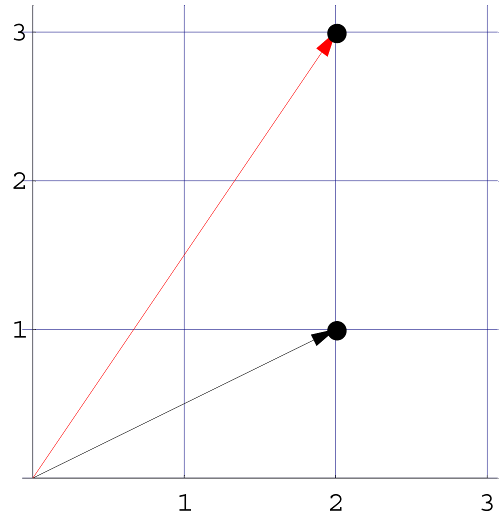
This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly ____________________ set.

Complete: A set of two vectors is linearly independent if_________________________________

________________________________.



linearly ____________________



linearly ____________________

3. ***A Set Containing the* 0 *Vector***

EXAMPLE 4 Consider the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} \right\}$.

A nontrivial solution to

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $x_1 =$ _____, $x_2 =$ _____, $x_3 =$ _____.

Theorem 9

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

4. *A Set Containing Too Many Vectors*

EXAMPLE 5 Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 7 & 6 & 5 & 4 \end{bmatrix}$. Explain why the columns of A are linearly dependent.

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

EXAMPLE 6 With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \end{bmatrix} \right\}$

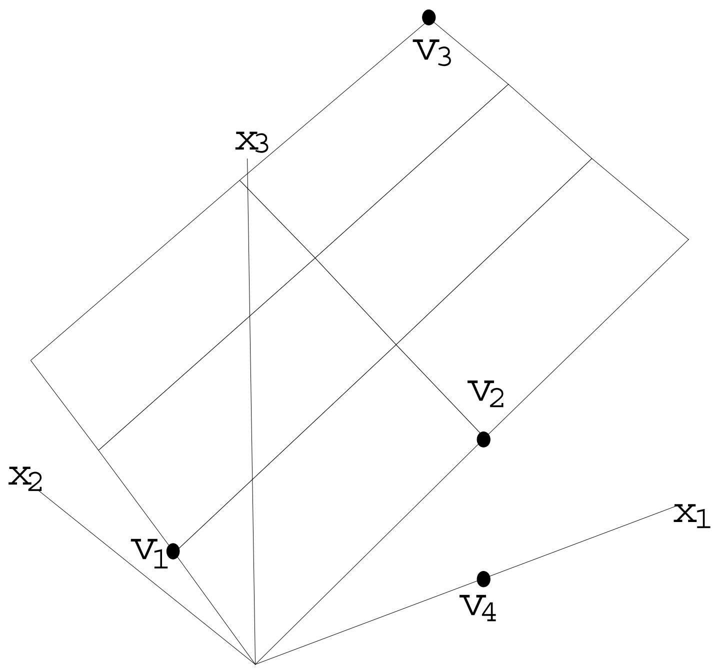
d. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

e. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

f. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$.

Characterization of Linearly Dependent Sets

EXAMPLE 7 Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain



Theorem 7

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

EXAMPLE With the least amount of work possible, decide if the following set of vectors is linearly independent.

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$