§ 3.3 Cramer's Rule, Volume and Linear Transformations

A method for solving linear systems

$$Ax = b.$$

Inefficient for large matrices.

If A is $n \times n$ and $b \in \mathbb{R}$, let $A_i(b)$ be the matrix obtained from a by replacing the ith column of A by b

$$A_i(b) = \begin{bmatrix} | & & | & & | \\ a_1 & \cdots & b & \cdots & a_n \\ | & & | & & | \end{bmatrix}$$

$$\uparrow$$

ith column.

Theorem 7 Cramer's Rule.

Let A be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution x of $Ax = b$ has entries

$$x_i = \frac{\det A_i(b)}{\det A}, \qquad i = 1, 2, \ldots, n.$$

Pf. Let A have columns $a_1, \ldots, a_n$
I_n — — — $e_1, \ldots, e_n$ (as usual).

Suppose x is s.t. $Ax = b$ (already know such an x exists and is unique) and let $I_i(x)$ be the matrix obtained from I_n by replacing e_i (column i) by x.

Then

$$A \cdot I_i(x) = A\,[e_1 \ldots x \ldots e_n]$$
$$= [Ae_1 \ldots Ax \ldots Ae_n]$$
$$= [a_1 \ldots b \ldots a_n] = A_i(b).$$

Taking determinants.

$$\det(A \cdot I_i(x)) = \det(A_i(b)).$$

$$\det(A)\det(I_i(x)) = \det(A_i(b)).$$

However, $\det I_i(x) = x_i$

- follows by cofactor expansion along column i

e.g. 3×3 case, $i = 2$.

$$\det A_2(x) = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = -x_1 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + x_2 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$+ x_3 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= x_3$$

Thus

$$\det A \cdot x_i = \det(A_i(b)).$$

and since A is inv., $\det A \neq 0$ and so

$$x_i = \frac{\det A_i(b)}{\det A}, \quad 1 \leq i \leq n. \quad \blacksquare$$

Ex. Use Cramer's Rule to solve

$Ax = b$

where

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{vmatrix}$$

cofactor exp. on first column.

$$= 5(15-8) - 2(9+8) + 0$$

$$= 5 \times 7 - 2 \times 17$$

$$= 35 - 34 = 1$$

$$\det A_1(b) = \det \begin{bmatrix} b & a_2 & a_3 \end{bmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ -2 & 5 & -2 \\ 3 & -4 & 3 \end{vmatrix}$$

$$= 1(15-8) - (-2)(9+8) + 3(-6-10)$$

$$= 7 + 2 \times 17 + 3 \times (-16)$$

$$= 7 + 34 - 48 = -7.$$

$$\det A_2(b) = \det\begin{bmatrix} a_1 & b & a_2 \end{bmatrix} = \begin{vmatrix} 5 & 1 & 2 \\ 2 & -2 & -2 \\ 0 & 3 & 3 \end{vmatrix}$$

$$= 5(-6+6) - 2(3-6) + 0$$

$$= 0 - 2(-3)$$

$$= 6$$

$$\det A_3(b) = \det\begin{bmatrix} a_1 & a_2 & b \end{bmatrix} = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{vmatrix}$$

$$= 5(15-8) - 2(9+4) + 0$$

$$= 5 \times 7 - 2 \times 13$$

$$= 35 - 26$$

$$= 9.$$

So $x_1 = \dfrac{\det A_1(b)}{\det A} = \dfrac{-7}{1} = -7$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{6}{1} = 6$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{9}{1} = 9$$

Answer $\quad x = \begin{bmatrix} -7 \\ 6 \\ 9 \end{bmatrix}$

Check

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} -7 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -35 + 18 + 18 \\ -14 + 30 - 18 \\ 0 - 24 + 27 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \checkmark$$

Cramer's Rule can be used to find the inverse A^{-1} of a matrix A.

Idea is to find vectors $x^{(1)}, \ldots, x^{(n)}$ s.t.

$$Ax^{(i)} = e_i \quad , \quad 1 \leq i \leq n$$

where
$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ — } i\text{ th row}$$

are as usual the std. basis vectors for $\mathbb{R}^n$.

Then if we let

$$D = \begin{bmatrix} | & & | \\ x^{(1)} & \cdots & x^{(n)} \\ | & & | \end{bmatrix}$$

$$AD = A\begin{bmatrix} | & & | \\ x^{(1)} & \cdots & x^{(n)} \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ Ax^{(1)} & \cdots & Ax^{(n)} \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix}$$

$$= I_n.$$

Thus, we have found D s.t. $AD = I_n$ and by k. $\Rightarrow$ a. of IMT (2.3 Th 8), A is invertible and $A^{-1} = D$.

In fact, we can be more explicit about the formula for A^{-1} than this.

<u>Thm</u> If A is an invertible $n \times n$ matrix, then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & & & \\ \vdots & & & \\ C_{1n} & & & C_{nn} \end{bmatrix}$$

Here $C_{ij} = (-1)^{i+j} \det A_{ij}$ is the ij cofactor of A.

Note that the matrix above is the <u>transpose</u> of the matrix of cofactors.

It is sometimes called the <u>adjugate matrix</u> of A, written $\operatorname{adj} A$, so

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A.$$